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Open the exam only after instruction by the assistants!

Partial Differential Equations (CES+SISC) | WS 2025/2026
Exam | 12.03.2026

Allowed resources:

- Use only permanent pens with blue or black ink. Particularly **no** red ink or pencil is allowed.
- Two hand-written, two-sided A4-papers in the original with name and matriculation number. No printed out papers are allowed.
- Other resources such as mobile phones, laptops etc. are prohibited.

Hint:

- Bringing resources which are specifically not allowed to possess at the seat in the exam is considered to be a cheating attempt.
- In total, you have **150 minutes** time to work on the exam. *All answers need to be explained sufficiently*.
- To pass the exam you need to have at least **50%** of the total points.
- The exam review takes place on 26.03.2026 starting at 10:00 – 12:00 in klPhys (1090|334). Appointments for the oral repeat-exam have to be arranged at the exam review.
- Please answer the questions starting on the page where the questions are posed. If you need additional space you can use the empty pages reserved at the end of the exam sheets. In this case please write your name and matriculation number on the respective pages as well as the question number.
- With your signature you confirm in all conscience that you feel well enough to take the exam and that you will not attempt cheating.

Matriculation number: ___ ___ ___ ___ ___ ___

Last name, first name: ______________________

Signature: ______________________

Task	1	2	3	4	5	6	7	8	$\sum$
Points	4	10.5	8.5	7	6	8	8	8	60
Your points									

Exam [ ] + Bonus [ ] = Total [ ]

Grade: [ ]

Problem 1.

For each of the following functions defined on Ω check if they are in $H^1(\Omega)$. Justify your answer. For each of the functions below that belong to $H^1(\Omega)$, compute the weak derivative **by applying the definition of the weak derivative**.

a) $\Omega = (0, 2)$,

$$u_1(x) = |x - 1|.$$

b) $\Omega = (0, 2)$,

$$u_2(x) = \begin{cases} x^2, & x < 0, \\ \cos(\pi x), & x > 0. \end{cases}$$

c) $\Omega = \mathbb{R}$,

$$u_3(x) = \begin{cases} (x+1)^2, & -1 < x < 0, \\ \sin(2\pi x), & 0 \leq x < 1, \\ 0, & \text{else.} \end{cases} .$$

2 + 1 + 1 = 4 Points

Name: Mat-Nr.:

Problem 2.

Let $\Omega \subset \mathbb{R}^2$ be open and bounded. The following boundary value problem is considered:

$$\begin{aligned} -\Delta u + u &= f \qquad \text{in } \Omega, \\ \nabla u \cdot n + \beta u &= g \qquad \text{on } \partial\Omega \end{aligned}$$

with $u \in H^1(\Omega)$, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ and $\beta \in \mathbb{R}$ with $\beta < 0$. n is the outer unit normal vector at $\partial\Omega$.

a) Derive the variational formulation.

b) Show continuity for the linear and the bilinear form of the variational formulation. Show coercivity for the bilinear form.

c) For which values of β has the variational formulation a unique solution? Why?

3 + 6.5 + 1 = 10.5 Points

Name: Mat-Nr.:

Problem 3.

Consider the following variational problem: Find $u \in H_0^1((0,1))$ such that

$$\int_0^1 u'v'dx = \int_0^1 v dx \qquad \forall v \in H_0^1((0,1)).$$

a) i) For general basis functions $\psi_i(x),\ 1 \leq i \leq N$, derive the Ritz-Galerkin equations for the variational problem and write down the general integral expressions for the stiffness matrix and the right-hand side.

ii) Compute the entries of the stiffness matrix for the specific basis functions $\psi_i(x) = x^{i+1} - x$.

b) We now consider the Finite Element method for the given variational problem. We use piecewise quadratic ansatz functions on an equidistant grid with mesh size h. Compute the entries $A_{1,1}, A_{2,3}$ and $A_{3,2}$ of the 3×3 element matrix for the element $[0,h]$. Ignore the boundary conditions.

Hint: The piecewise quadratic ansatz functions for a uniform mesh are given by

$$\phi_i^{\text{(inner element point)}} = \begin{cases} \frac{4(x-x_{i-1})(x_{i+1}-x)}{h^2} & x_{i-1} \leq x \leq x_{i+1} \\ 0 & \text{else} \end{cases}$$

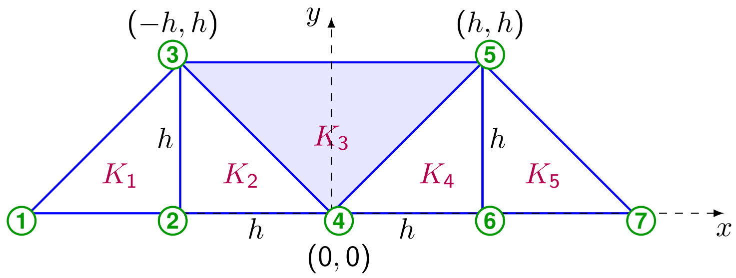
$$\phi_i^{\text{(boundary element point)}} = \begin{cases} \frac{2(x_{i-2}-x)(x_{i-1}-x)}{h^2} & x_{i-2} \leq x \leq x_i \\ \frac{2(x-x_{i+1})(x-x_{i+2})}{h^2} & x_i \leq x \leq x_{i+2} \\ 0 & \text{else.} \end{cases}$$

3 + 2 + 3.5 = 8.5 Points

Name: Mat-Nr.:

Problem 4.

a) We consider the following triangulation of a domain Ω. On this mesh, we use the



space of linear finite elements, that is, the test and basis functions are continuous, and on each triangle, the linear ansatz function $\psi(x, y) = \alpha x + \beta y + \gamma$ is used, with $\alpha, \beta, \gamma \in \mathbb{R}$.

i) On the triangle K_3, find the two linear form/shape functions N_1, N_2 corresponding to the nodes 3 and 4 such that

$$N_1(-h, h) = 1,\ N_1(0, 0) = 0,\ N_1(h, h) = 0$$

and

$$N_2(-h, h) = 0,\ N_2(0, 0) = 1,\ N_2(h, h) = 0$$

ii) Determine which elements K_i contribute to the entry $a(\varphi_3, \varphi_4)$ of the global stiffness matrix, where φ_3, φ_4 are the piecewise linear hat-functions corresponding to nodes 3 and 4.

iii) Which elements of the global stiffness matrix $\boldsymbol{A}$ given by $A_{ij} = a(\varphi_i, \varphi_j)$ are zero? To answer this question, set up a matrix and fill it using 0 for zeros and $\times$ for non-zero entries. φ_k are again piecewise linear hat-functions.

> **Hint:** The entries of the global stiffness matrix are computed by
>
> $$A_{ij} = a(\varphi_i, \varphi_j) = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j dx$$

b) Consider the ansatz function

$$\psi(x, y) = \alpha x + \beta y + \gamma(1 - x - y) + \delta xy(1 - x - y),\ \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

on the triangle with vertices $p_1 = (0, 0), p_2 = (1, 0)$ and $p_3 = (0, 1)$. Show that given the values $u_1, u_2, u_3, u_4 \in \mathbb{R}$ and the additional point $p_4 = (\frac{1}{2}, \frac{1}{2})$, a unique function ψ of the above form **cannot** be found with

$$\psi(p_i) = u_i,\ i = 1, 2, 3, 4.$$

2 + 1 + 2 + 2 = 7 Points

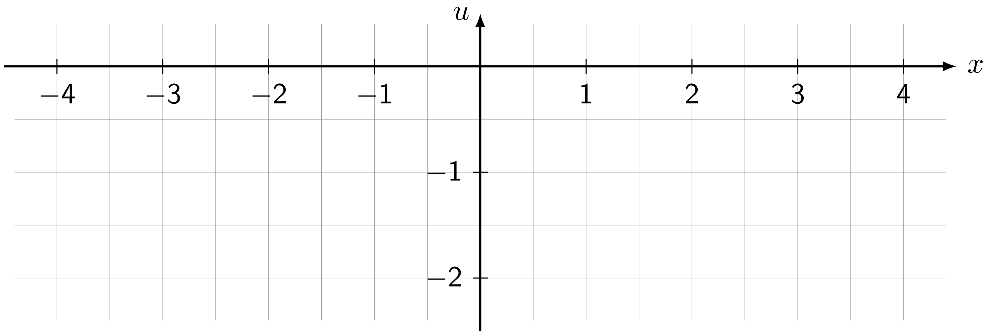
Name: Mat-Nr.:

Problem 5.

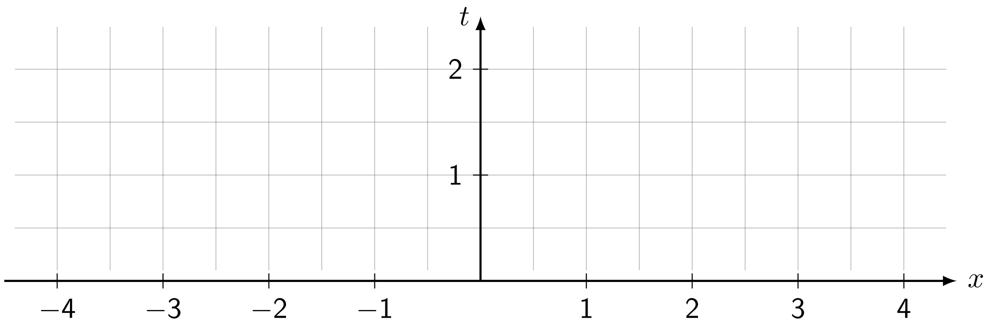
For $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, we consider the Burgers' equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0, \quad u(x,0) = \begin{cases} -1, & x \le -1, \\ -\frac{x}{2} - \frac{3}{2}, & -1 < x < 1, \\ -2, & x \ge 1. \end{cases}.$$

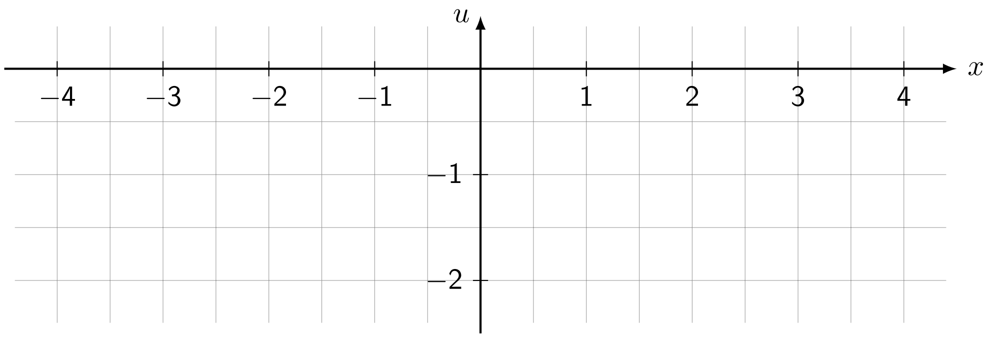
a) Draw the initial condition $u(x,0)$ into the coordinate system below.



b) Draw the characteristics in x–t-plane through the points $(x_0, 0)$ with $x_0 = -2, -1, 0, 1, 2, 3, 4$ up to time $t = 2$.



c) Draw the exact solution at time $t = 2$.



d) Compute the shock speed s for the jump discontinuity appearing at time $t = 2$.

1 + 1.5 + 1.5 + 2 = 6 Points

Name: Mat-Nr.:

Problem 6.

For $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ consider the following system of linear conservation laws

$$\partial_t U(t,x) + A\partial_x U(t,x) = 0,$$

with matrix $A = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$ and initial condition

$$U(0,x) = \begin{cases} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & x < 0, \\ \begin{pmatrix} 3 \\ -1 \end{pmatrix} & x > 0. \end{cases}$$

Derive the solution by performing the following steps:

a) Diagonalize the system and transform the initial condition to reduce the problem into two independent scalar linear conservation laws by changing variables. For this purpose use the eigenvalue decomposition of the matrix $A = T\Lambda T^{-1}$ with a diagonal matrix Λ.

b) Solve the acquired independent scalar conservation laws with corresponding initial conditions.

c) Transform the solutions back to the variable $U(t,x)$.

3 + 2 + 3 = 8 Points

Name: Mat-Nr.:

Problem 7.

For $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, we consider the linear advection equation

$$\partial_t u + a\partial_x u = 0\,, \quad a \in \mathbb{R},$$

We will discretize the above equation using finite differences and a Lax-Friedrichs scheme in the following way

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) + \frac{a\lambda}{2}(u_{j-1}^n - u_{j+1}^n).$$

where u_j^n represents the solution at the j-th grid point and the n-th time step. The factor λ is the ratio of Δt and Δx i.e. $\lambda = \Delta t/\Delta x$.

a) Show that the above LF scheme is consistent up to at least first order.

b) Let $g(\theta)$ be the amplification factor of the scheme.
Using the Fourier ansatz $u_j^n = e^{\mathrm{i}k\pi x_j} = e^{\mathrm{i}jk\pi\Delta x}$ show that

$$g(\theta) = \cos(\theta) - \mathrm{i}a\lambda\sin(\theta),$$

where $\theta = k\pi\Delta x$ with k being the wave number of the Fourier ansatz.

c) For what ranges of λ is the scheme stable?

d) For what ranges of λ is the scheme monotone?

2 + 2 + 2 + 2 = 8 Points

Name: Mat-Nr.:

Problem 8.

The *super-bee limiter* for the nonlinear reconstruction of cell slopes in a finite volume method is given by

$$\phi^{(\mathrm{sb})}(\theta) = \max\left(0, \min\left(1, 2\theta\right), \min\left(2, \theta\right)\right)$$

(a) Draw the super-bee limiter in the θ-ϕ-diagram together with the minmod and van-Leer limiter.

(b) Is the superbee-limiter consistent and TVD?

(c) Consider the reconstruction formula $\widetilde{u}_i(x) = u_i + \sigma_i(x - x_i)$ with $\sigma_i = \phi(\theta_i)\frac{u_{i+1}-u_i}{\Delta x}$ and the values

1. $u_{i-1} = 3, \quad u_i = 4, \quad u_{i+1} = 10$ and
2. $u_{i-1} = 2, \quad u_i = 0, \quad u_{i+1} = 6.$

Compute the slope σ_i based on the super-bee and minmod limiter.

2 + 2 + 4 = 8 Points

Name: Mat-Nr.: