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**Open the exam only after instruction by the assistants!**

# Partial Differential Equations (CES+SISC) | WS 2024
# Exam | 05.03.2025

**Allowed resources:**

- Use only permanent pens with blue or black ink. Particularly **no** red ink or pencil is allowed.
- Two hand-written, two-sided A4-papers in the original with name and matriculation number. No printed out papers are allowed.
- Other resources such as mobile phones, laptops etc. are prohibited.

**Hint:**

- Bringing resources which are specifically not allowed to possess at the seat in the exam is considered to be a cheating attempt.
- In total, you have **150 minutes** time to work on the exam. *All answers need to be explained sufficiently*.
- To pass the exam you need to have at least **50%** of the total points.
- The exam review takes place on 20.03.2025 starting at 14:00 – 16:00 in klPhys (1090|334). Appointments for the oral repeat-exam have to be arranged at the exam review.
- Please answer the questions starting on the page where the questions are posed. If you need additional space you can use the empty pages reserved at the end of the exam sheets. In this case please write your name and matriculation number on the respective pages as well as the question number.
- With your signature you confirm in all conscience that you feel well enough to take the exam and that you will not attempt cheating.

**Matriculation number:** ___ ___ ___ ___ ___ ___

**Last name, first name:** ______________________

**Signature:** ______________________

| Task | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\sum$ |
|---|---|---|---|---|---|---|---|---|---|
| Points | 5 | 8 | 8 | 9 | 6 | 8 | 8 | 8 | 60 |
| Your points | | | | | | | | | |

Exam [ ] + Bonus [ ] = Total [ ]

Grade: [ ]

**Problem 1.**

We consider periodic, real-valued functions $u$ on the intervall $[-1,1] \subset \mathbb{R}$. They can be written as Fourier series

$$u(x) = \sum_{k\in\mathbb{Z}} \alpha_k e^{\mathbf{i}k\pi x} \qquad x \in [-1,1]$$

with Fourier coefficients $\alpha_k \in \mathbb{C}$. The integral satisfies

$$\int_{-1}^{1} |u(x)|^2 \,\mathrm{d}x = \sum_{k\in\mathbb{Z}} |\alpha_k|^2$$

(a) First, consider the $H^1$-function $u : [a,b] \to \mathbb{R}$. What is the definition of the Sobolev-norm $\|u\|_{H^1([a,b])}$?

(b) Show that for periodic, real-valued functions $u$ on $[-1,1]$ we have the relation

$$\|u\|_{H^1([-1,1])} = \sqrt{\sum_{k\in\mathbb{Z}} (1 + k^2\pi^2) |\alpha_k|^2}$$

for the Sobolev-norm.

(c) Show that the one-dimensional Poisson problem

$$-\Delta u = f, \qquad x \in [-1,1]$$

for a periodic function $f$ has the periodic solution

$$u(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\beta_k}{k^2\pi^2} e^{\mathbf{i}k\pi x},$$

where $\beta_k$, $k \in \mathbb{Z}$ are the Fourier coefficients of the given function $f$.

**Hint:** You can compare the Fourier coefficients on both sides of the equation due to orthogonality.

**Remark:** We ignored the coefficient at $k = 0$ assuming a zero mean of $u$.

(d) Compute the $H^1$-norm of the Poisson solution $u$ from (c) in terms of its Fourier coefficients and proof the relation

$$\|u\|_{H^1([-1,1])} \leq \|f\|_{L^2([-1,1])}$$

for periodic solutions.

**1 + 2 + 1 + 1 = 5 Points**

**Problem 2.**

On the domain $\Omega \subset \mathbb{R}^2$ we look for the unknown functions $u : \Omega \to \mathbb{R}$ and $\boldsymbol{s} : \Omega \to \mathbb{R}^2$ given by the system of partial differential equations

$$\begin{cases} \operatorname{div}(\boldsymbol{s}) &= -u + f \\ \operatorname{grad}(u) &= -\boldsymbol{s} \end{cases} \qquad \text{in } \Omega,$$
$$\mathbf{s} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega,$$

where $\boldsymbol{n}$ is the outer normal of the domain $\Omega$ and $f : \Omega \to \mathbb{R}$ a given source.

(a) Eliminate the variable $s$ and find an alternative PDE for $u$.

(b) By using the test functions $(v, \boldsymbol{r})^T$ derive the variational formulation or weak form

$$-\int_\Omega \boldsymbol{s} \cdot \operatorname{grad}(v)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{r} \cdot \operatorname{grad}(u)\,\mathrm{d}\boldsymbol{x} + \int_\Omega v\,u\,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{r} \cdot \boldsymbol{s}\,\mathrm{d}\boldsymbol{x} = \int_\Omega v\,f\,\mathrm{d}\boldsymbol{x} \quad \forall \begin{pmatrix} v \\ \boldsymbol{r} \end{pmatrix}$$

of the original system. Give a formal reason for the choice $u, v \in H^1(\Omega)$ and $\boldsymbol{s}, \boldsymbol{r} \in L^2(\Omega)$. Write the variational formulation as

$$a\left(\begin{pmatrix} u \\ \boldsymbol{s} \end{pmatrix}, \begin{pmatrix} v \\ \boldsymbol{r} \end{pmatrix}\right) = l\left(\begin{pmatrix} v \\ \boldsymbol{r} \end{pmatrix}\right)$$

and identify $a$ and $l$.

(c) Show that the bilinear form $a$ is ***not*** symmetric.

(d) Show that the statement

$$\left| a\left(\begin{pmatrix} u \\ \boldsymbol{s} \end{pmatrix}, \begin{pmatrix} u \\ \boldsymbol{s} \end{pmatrix}\right) \right| = \|u\|^2_{L^2(\Omega)} + \|\boldsymbol{s}\|^2_{L^2(\Omega)}$$

holds. Is the bilinear form coercive for functions $\begin{pmatrix} u \\ \boldsymbol{s} \end{pmatrix} \in H^1(\Omega) \times L^2(\Omega)$?

**Hint:** We define the norm

$$\left\| \begin{pmatrix} u \\ \boldsymbol{s} \end{pmatrix} \right\|_{H^1(\Omega) \times L^2(\Omega)} := \sqrt{\|u\|^2_{H^1(\Omega)} + \|\boldsymbol{s}\|^2_{L^2(\Omega)}}.$$

**2 + 3 + 1 + 2 = 8 Points**

Name: Mat-Nr.:

**Problem 3.**

Let $V$ be a Hilbert space and consider a bilinear form $a : V \times V \to \mathbb{R}$ as well as a linear form $b : V \to \mathbb{R}$ that satisfy the conditions of the Lax-Milgram theorem. Let also $V_N$ be an $N$-dimensional subspace of $V$.

(a) Derive a linear system of equations for the Ritz-Galerkin solution $u_N \in V_N$ of the variational formulation in $V$ defined by $a$ and $b$.

(b) Which properties does the Ritz-Galerkin matrix $A_N$ have? Is the linear Ritz-Galerkin system uniquely solvable?

(c) Consider the following minimization problem:

$$\min_{v \in V} J(v), \qquad \text{with } J(v) = \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} - \int_\Omega v \,\mathrm{d}x,$$

where $V = H^1_0(\Omega)$ and $\Omega = (0,\pi) \times (0,\pi) \subset \mathbb{R}^2$. What is the corresponding PDE to this minimization problem and weak formulation. Use the $M$-dimensional subspace $V_M$ spanned by the following basis functions:

$$\psi_{i,j}(x,y) = \sin(ix)\sin(jy), \qquad 1 \le i,j \le N, \qquad M = N^2,$$

to derive elements of the corresponding Ritz-Galerkin stiffness matrix and right hand side.

**Hint:** You can use without a proof the following integrals:

$$\int_0^\pi \sin(kx) = \frac{-1+\cos(k\pi)}{k}, \quad \text{for } k \in \mathbb{N},$$

$$\int_\Omega \begin{pmatrix} i\cos(ix)\sin(jy) \\ j\cos(jy)\sin(ix) \end{pmatrix} \begin{pmatrix} k\cos(kx)\sin(ly) \\ l\cos(ly)\sin(kx) \end{pmatrix} \mathrm{d}\Omega = \begin{cases} \frac{(i^2+j^2)\pi^2}{4}, & (i,j) = (k,l) \\ 0, & \text{otherwise} \end{cases}$$

**3 + 1 + 4 = 8 Points**

Name: Mat-Nr.:

**Problem 4.**

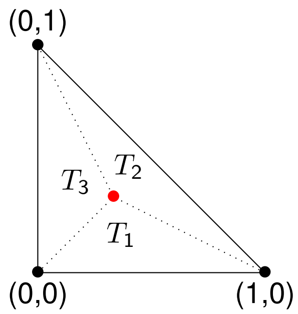
Given the boundary value problem

$$-\left(\left(1+x^2\right)u_x\right)_x-\left(\left(1+y^2\right)u_y\right)_y+2u=2 \qquad \text{in } T$$
$$u=0 \qquad \text{on } \partial T$$

on the reference triangle $T := \{(x,y) : 0 < x, y < 1,\ x+y < 1\}$ with the vertices $(0,0)$, $(1,0)$ and $(0,1)$.

(a) Give the weak formulation of the problem.

(b) We want to solve the problem using piecewise linear finite elements. For this reason $T = T_1 \cup T_2 \cup T_3$ is split up into three triangles $T_i$ with the common inner vertex $(x_0, y_0) = (1/3, 1/3)$. Set up the linear system for the approximate solution $u_h$ but leave the matrix and vector elements in integral form.




> **Hint:** What does the homogeneous boundary mean for the number of basis functions?
> On each triangle $T_i$ the basis function is given by $\varphi_i(x,y) = \alpha_i x + \beta_i y + \gamma_i,\ \alpha_i, \beta_i, \gamma_i \in \mathbb{R},\ i = 1, \ldots, 3$.

(c) Find the transformations $A_i : T \to T_i,\ i = 1, \ldots, 3$ from the reference triangle onto each of the three triangles.The transformation should map the common vertex $(x_0, y_0)$ onto the origin $(0,0)$.

(d) Determine the integrals of the equation system. Use the transformations $A_i$ derived in the task before and the integral transformation

$$\int_{T_i} f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_T f(A_i(\eta,\xi))\ |\det A_i(\eta,\xi)|\ \mathrm{d}\eta\mathrm{d}\xi$$

to calculate the integrals over the reference triangle.

> **Hint:** Make yourself clear how $\varphi_i$ and $\nabla\varphi_i$ transform.
> For the integrals over the reference triangle $T$ you can use the following quadrature rule
> $$\int_T f(x,y)\,\mathrm{d}x\mathrm{d}y = \frac{1}{6}\left(f\left(\frac{1}{6},\frac{1}{6}\right)+f\left(\frac{4}{6},\frac{1}{6}\right)+f\left(\frac{1}{6},\frac{4}{6}\right)\right).$$

(e) Determine the approximate solution by solving the linear system. What is the approximate value at the position $(x_0, y_0)$?

**1 + 1.5 + 1.5 + 4 + 1 = 9 Points**

Name: Mat-Nr.:

**Problem 5.**

Consider the two PDEs as follows

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \tag{1}$$

$$\left(u^2\right)_t + \left(\frac{2u^3}{3}\right)_x = 0. \tag{2}$$

a) Show that (1) and (2) have the same classical solutions.

b) Consider Riemann problem with the initial data

$$u(x, t = 0) = u_0(x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \quad \text{where} \quad u_L > u_R.$$

(i) Determine weak solutions for both (1) and (2).

(ii) Are both weak solutions in (i) the same?

**2 + 4 = 6 Points**

Name: Mat-Nr.:

**Problem 6.**

For a following system of linear conservation laws

$$\partial_t U(t,x) + A\partial_x U(t,x) = 0,$$

with a matrix $A = \begin{pmatrix} -7 & 10 \\ 5 & -2 \end{pmatrix}$ and an initial condition

$$U(0,x) = \begin{cases} \begin{pmatrix} 0 \\ 4 \end{pmatrix}, & x < 0, \\ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, & x > 0. \end{cases}$$

Consider following steps:

(a) Diagonalise the system and transform the initial condition to reduce it to two independent scalar linear conservation laws by changing variables. For this purpose use the eigenvalue decomposition of the matrix $A = T\Lambda T^{-1}$ with a diagonal $\Lambda$.

(b) Solve acquired independent scalar conservation laws with corresponding initial conditions.

(c) Transform the solution back to the variable $U(t,x)$.

**4 + 1 + 3 = 8 Points**

Name: Mat-Nr.:

**Problem 7.**

For the advection equation

$$\partial_t u + a\partial_x u = 0 \tag{$\star$}$$

with periodic boundaries we consider an equidistant discretization with time step $\Delta t$ and grid size $\Delta x$, and the time-step-method

$$u_j^{n+1} = u_j^n + \frac{\nu}{2}\left(u_{j-1}^n - u_{j+1}^n\right) + \frac{\delta}{2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right) \tag{$\star\star$}$$

for point values $u_j^n \approx u(x_j, t_n)$. The Courant number is given by $\nu = \frac{a\Delta t}{\Delta x}$ and there is an additional parameter $\delta \in \mathbb{R}$.

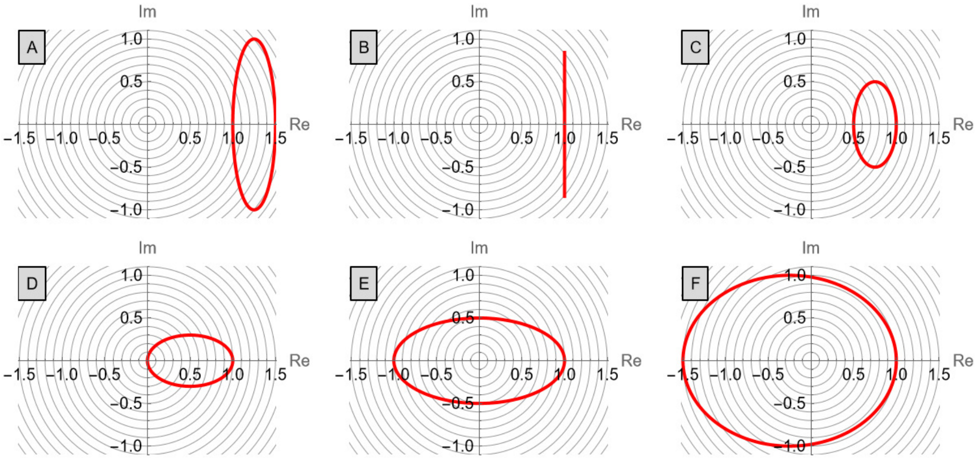
(a) For general $\delta$, compute the local error and consistency order for the scheme ($\star\star$). What value of $\delta$ is needed to make the scheme second-order?

(b) For general $\delta$ show that amplification function of the method is given by

$$g(\xi) = 1 + \delta\left(\cos\xi - 1\right) - \mathbf{i}\,\nu\sin\xi$$

based on the Fourier or von-Neumann analysis.

(c) Consider the following 6 parametric plots of the amplification function $g(\xi)$ in the complex plane.




What are the values of the parameters $\nu$ and $\delta$ for each of these plots? In what cases do you expect a stable method in the sense of von-Neumann? Please justify!

(d) Name the methods for $\delta = \nu$ and $\delta = \nu^2$.

**3 + 2 + 2 + 1 = 8 Points**

Name: Mat-Nr.:

**Problem 8.**

The *super-bee limiter* for the nonlinear reconstruction of cell slopes in a finite volume method is given by

$$\phi^{(\mathsf{sb})}(\theta) = \max\left(0, \min\left(1, 2\theta\right), \min\left(2, \theta\right)\right)$$

(a) Draw the super-bee limiter in the $\theta$-$\phi$-diagram together with the minmod and van-Leer limiter.

(b) Is the superbee-limiter consistent and TVD?

(c) Consider the reconstruction formula $\widetilde{u}_i(x) = u_i + \sigma_i(x - x_i)$ with $\sigma_i = \phi(\theta_i)\frac{u_{i+1} - u_i}{\Delta x}$ and the values

1. $u_{i-1} = 3, \quad u_i = 4, \quad u_{i+1} = 10$ and
2. $u_{i-1} = 2, \quad u_i = 0, \quad u_{i+1} = 6.$

Compute the slope $\sigma_i$ based on the super-bee and minmod limiter.

**3 + 2 + 3 = 8 Points**

Name: Mat-Nr.: