Prof. Dr. M. Torrilhon
Dr. Michael Redle
Vladimir Dordic

Applied and Computational Mathematics | RWTH AACHEN UNIVERSITY

Open the exam only after instruction by the assistants!

Partial Differential Equations (CES+SISC) | WS 2023/24
Exam | 08.03.2024

Allowed resources:

- Use only permanent-ink pens and no red-ink (or similar).
- Two hand-written, two-sided A4-papers with name and matriculation number.
- Other resources such as mobile phones, laptops etc. are not allowed.

Hint:

- Bringing resources which are specifically not allowed to possess at the seat in the exam is considered to be a cheating attempt.
- In total, you have **150 minutes** time to work on the exam.
  All answers need to be explained sufficiently.
- To pass the exam you need to have at least **50%** of the total points.
- The exam review takes place on 22.03.2024 starting at 14:00 – 15:00 in Kleine Physik (klPhys) (1090|334). Appointments for the oral repeat-exam have to be arranged at the exam review.
- Please answer the questions starting on the page where the questions are posed. If you need additional space you can use the empty pages reserved at the end of the exam sheets. In this case please write your name and matriculation number on the respective pages as well as the question number.
- With your signature you confirm in all conscience that you feel well enough to take the exam and that you will not attempt cheating.

Matriculation number: ___ ___ ___ ___ ___ ___

Last name, first name: ____________________

Signature: ____________________

Task	1	2	3	4	5	6	7	8	$\sum$
Points	6	7	8	9	8	9	7	6	60
Your points									

Exam		Bonus		Total
	+		=	

Grade:

Aufgabe 1.

For each of the following functions check whether it is in $H^1(\mathbb{R})$ or not. Justify your answers. Calculate the weak derivative for the functions that are in $H^1(\mathbb{R})$.

(a)

$$u_1 = \begin{cases} (x+2)^2, & -2 < x < 0, \\ (x-2)^2, & 0 \leq x < 2, \\ 0, & \text{otherwise}, \end{cases}$$

(b)

$$u_2(x) = \begin{cases} x & 0 \leq x \leq 1, \\ x+2 & 1 < x < 2, \\ 0, & \text{otherwise}, \end{cases}$$

(c)

$$u_3 = \begin{cases} \cos(x), & x \geq \frac{\pi}{2}, \\ 0, & \text{otherwise}. \end{cases}$$

2 + 2 + 2 = 6 Points

Name: Mat-Nr.:

Aufgabe 2.

Let $\Omega \subset \mathbb{R}^2$ be open and bounded. The boundary value problem is considered

$$\begin{aligned} -\Delta u + \alpha u &= f \quad \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with $u \in H^1(\Omega)$ and $f \in L^2(\Omega)$, $\alpha \in \mathbb{R}$ and $\alpha > 0$.

(a) Derive the weak formulation of the above problem.

(b) Derive the bilinear form resulting from the boundary value problem. Show that it is coercive in the $H^1(\Omega)$-norm.

(c) Is the linear form describing the right-hand side continuous? Justify your answer.

2 + 2 + 3 = 7 Points

Aufgabe 3.

Consider the boudary value problem

$$\begin{aligned} -\Delta u(x,y) &= 1 \quad \text{in } \ \Omega = (0,1)^2 \subset \mathbb{R}^2, \\ u(x,y) &= 0 \quad \text{on } \ \partial\Omega. \end{aligned}$$

Choose the basis functions as

$$\psi_{i,j}(x,y) = \sin(i\pi x)\sin(j\pi y), \quad 1 \le i,j \le N.$$

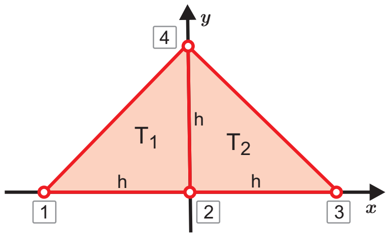
(a) Derive the Ritz-Galerkin equations and find the stiffness matrix.

(b) Let basis functions $\psi_{1,1}, \psi_{1,3}, \psi_{3,1}$ and $\psi_{3,3}$ form a space V. Find the approximate solution in the space V.

4 + 4 = 8 Points

Aufgabe 4.

For $\Omega \subset \mathbb{R}^2$ we consider a subset of a triangulation that consists of four nodes P_k, $k = 1, 2, 3, 4$, and two triangles given by (P_1, P_2, P_4) and (P_2, P_3, P_4). The edges along the x- and y-axis are of length $h \in \mathbb{R}$, and the coordinates of the node P_2 are $(0, 0)$.



On each triangle we have a linear ansatz function

$$p(x, y) = \alpha + \beta\, x + \gamma\, y$$

with parameters $\alpha, \beta, \gamma \in \mathbb{R}$. As numerical parameters we use the point values at the nodes $u_k = u(P_k)$, $k = 1, 2, 3, 4$.

(a) Give the explicit form of the linear functions in the triangles $T_{1,2}$ for given values of u_k, $k = 1, 2, 3, 4$.

(b) The bilinear form $a(u, v)$ is being evaluated using piece-wise linear hat functions φ_k on each vertex P_k, $k = 1, 2, 3, 4$. Which entries of the submatrix

$$A_{jk} = a\left(\varphi_j, \varphi_k\right), \quad j, k = 1, 2, 3, 4$$

will be zero independent of a?

(c) When evaluating the bilinear form, the underlying integrals are typically computed element-wise using element-form-functions $N_j^{(T)}(x, y)$. Use the result of (a) and list the four element-form-functions for the nodes $j \in \{2, 4\}$ and the triangle T_1 and T_2.

(d) Compute the entry A_{24} of the matrix in (b) for the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

with the result of (c). No transformation to the reference-element necessary.

3 + 2 + 2 + 2 = 9 Points

Name: Mat-Nr.:

Aufgabe 5.

We consider the *homogeneous shallow flow equations* given as follows

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\Big(hu^2 + \frac{1}{2}\left(gh^2\right)\Big) = 0, \end{cases}$$

with g being a positive parameter, so-called gravitational acceleration. We write the two equations as a system

$$\partial_t U + \partial_x F(U) = 0.$$

a) Define U, and $F(U)$.

b) We linearize the given system as the following expression

$$\partial_t U + A(U)\partial_x U = 0.$$

Compute the flux *Jacobian* $A(U) = DF(U)$.

c) What are the characteristic velocities of the system?

d) Under which condition is the system hyperbolic?

2 + 2 + 2 + 2 = 8 Points

Name: Mat-Nr.:

Aufgabe 6.

Consider the linear system of conservation laws

$$\partial_t U + A \partial_x U = 0 \tag{\#}$$

with matrix $A = \begin{pmatrix} 1 & 1 \\ \varepsilon^2 & 1 \end{pmatrix}$ using $\varepsilon > 0$ for a solution vector $U : \mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}^2$.

(a) Show that the system is hyperbolic for all $\varepsilon > 0$. What happens to hyperbolicity in the special case $\varepsilon = 0$?

(b) Show that $v_{1,2} = (\pm\frac{1}{\varepsilon}, 1)^T$ are the eigenvectors of the matrix A. What are the eigenvalues?

(c) Diagonalize the hyperbolic system $(\#)$ for general $\varepsilon > 0$.

(d) Consider the initial conditions

$$U(x,0) = \begin{cases} \begin{pmatrix} 2 \\ 2 \end{pmatrix}, & x < 0 \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & x > 0 \end{cases}$$

for the system $(\#)$. What is the resulting intermediate state $U^\star$ of the corresponding Riemann solution for general $\varepsilon > 0$.

(e) Give a sketch of the Riemann solution for small ε. Why is the solution called *"delta-shock"* in the limit $\varepsilon \to 0$?

2 + 2 + 1 + 2 + 2 = 9 Points

Name: Mat-Nr.:

Aufgabe 7.

Consider the following advection equation

$$\partial_t u + a\partial_x u = 0 \qquad \forall x \in \Omega.$$

We will discretize the above equation using finite differences and a Lax-Friedrichs scheme in the following way

$$u_i^{n+1} = \frac{1}{2}(u_{i-1}^n + u_{i+1}^n) + \frac{a\lambda}{2}(u_{i-1}^n - u_{i+1}^n).$$

where u_i^n represents the solution at the i-th grid point and the n-th time step. The factor λ is the ratio of Δt and Δx i.e. $\lambda = \Delta t/\Delta x$. Show that

a) The above LF scheme is consistent upto at least first order.

b) Let $g(\theta)$ be the amplification factor of the scheme, then

$$g(\theta) = \cos(\theta) - \mathrm{i}a\lambda\sin(\theta),$$

where $\theta = k\pi\Delta x$ with k being the wave number of the Fourier ansatz.

c) For what ranges of λ is the scheme stable?

3 + 2 + 2 = 7 Points

Name: Mat-Nr.:

Aufgabe 8.

Consider the following numerical schemes for the linear advection equation

$$u_t + au_x = 0, \quad \text{with} \quad a > 0.$$

Rewrite the scheme in the incremental form as follows

$$u_i^{n+1} = u_i^n + C_{i+1/2}^n \left(u_{i+1}^n - u_i^n\right) - D_{i-1/2}^n \left(u_i^n - u_{i-1}^n\right).$$

Compute the corresponding $D_{i-1/2}^n$ and $C_{i+1/2}^n$, and comment on the TVD property.

a) The Upwind scheme with the correct left hand-side stencil (as $a > 0$):

$$u_i^{n+1} = u_i^n - c\left(u_i^n - u_{i-1}^n\right).$$

b) The Upwind scheme with the right-hand side stencil:

$$u_i^{n+1} = u_i^n - c\left(u_{i+1}^n - u_i^n\right).$$

c) The *Lax-Friedrich* scheme:

$$u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{c}{2}\left(u_{i+1}^n - u_{i-1}^n\right).$$

Note in passing that $c = \dfrac{a\Delta t}{\Delta x}$ is the CFL number.

2 + 2 + 2 = 6 Points

Name: Mat-Nr.: