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Applied and Computational Mathematics | RWTH AACHEN UNIVERSITY

Open the exam only after instruction by the assistants!

Partial Differential Equations (CES+SISC) | WS 2022/23
Exam | 06.03.2023

Allowed resources:

- Use only permanent-ink pens and no red-ink (or similar).
- Two hand-written, two-sided A4-papers with name and matriculation number.
- Other resources such as mobile phones, laptops etc. are not allowed.

Hint:

- Bringing resources which are specifically not allowed to possess at the seat in the exam is considered to be a cheating attempt.
- In total, you have **150 minutes** time to work on the exam.
  All answers need to be explained sufficiently.
- To pass the exam you need to have at least **50%** of the total points.
- The exam review takes place on 20.03.2023 starting at 10:00 – 12:00 in Kleine Physik (klPhys) (1090|334). Appointments for the oral repeat-exam have to be arranged at the exam review.
- Please answer the questions starting on the page where the questions are posed. If you need additional space you can use the empty pages reserved at the end of the exam sheets. In this case please write your name and matriculation number on the respective pages as well as the question number.
- With your signature you confirm in all conscience that you feel well enough to take the exam and that you will not attempt cheating.

Matriculation number: ___ ___ ___ ___ ___ ___

Last name, first name: ______________________

Signature: ______________________

Task	1	2	3	4	5	6	7	8	$\sum$
Points	6	10	7	7	7	6	9	8	60
Your points									

Exam		Bonus		Total
	+		=	

Grade:

Aufgabe 1.

For each of the following three functions defined on Ω, check whether they are in $H^1(\Omega)$. If yes, compute the weak derivative. Also discuss whether they are in $C^1(\Omega)$:

- $u_1(x) = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}, \quad \Omega = \mathbb{R}$
- $u_2(x) = \begin{cases} 1 + x & x < 0 \\ 1 - x & x \geq 0 \end{cases}, \quad \Omega = [-1, 1]$
- $u_3(x) = \begin{cases} (x+1)^2 & x < 0 \\ (x-1)^2 + 1 & x \geq 0 \end{cases}, \quad \Omega = \mathbb{R}$

2 + 2 + 2 = 6 Points

Name: Matriculation-Nr.:

Aufgabe 2.

Consider the following problem with the Neumann boundary condition:

$$\begin{cases} -\alpha\Delta u + \beta u = f(x) & \text{in } \Omega := (0,1)\times(0,1) \\ \partial_n u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f(x) = x_2^4$ if $x = (x_1, x_2)$ and $\alpha \neq 0,\ \beta > 0$ are real numbers.

(a) Derive the weak formulation of the above problem.

(b) Show existence and uniqueness of the weak solution.

(c) Comment on the existence of weak solutions for $\beta = 0$.

(d) Which (simpler) form does the PDE take for $\beta = \alpha$?

3 + 4 + 2 + 1 = 10 Points

Name: Matriculation-Nr.:

Aufgabe 3.

Consider the boundary value problem

$$\begin{aligned} \Delta u(x,y) &= 1 \quad \text{in } \ \Omega = (0,1)^2 \subset \mathbb{R}^2 \\ u(x,y) &= 0 \quad \text{on } \ \partial\Omega \end{aligned}$$

Choose the basis functions as

$$\psi_{i,j}(x,y) = \sin(i\pi x)\sin(j\pi y), \quad 1 \le i,j \le N$$

(a) Derive the weak formulation of the problem.

(b) Find the Ritz-Galerkin equations and the stiffness matrix.

(c) Let basis functions $\psi_{1,1}, \psi_{1,2}, \psi_{2,1}$ and $\psi_{2,2}$ form a space V. Find the approximate solution in the space V.

Hint:

$$\int_\Omega (\nabla\psi_{i,j} \cdot \nabla\psi_{k,l})\, d\Omega = \begin{cases} \frac{(i^2+j^2)\pi^2}{4}, & (i,j) = (k,l) \\ 0, & \text{otherwise} \end{cases}$$

$$\int_\Omega \psi_{k,l}\, d\Omega = \begin{cases} \frac{4}{kl\pi^2}, & \text{neither } k \text{ nor } l \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

1 + 3 + 3 = 7 Points

Name: Matriculation-Nr.:

Aufgabe 4.

For $\Omega \subset \mathbb{R}^2$, we consider a uniform quadrilateral grid that is build from squares of equal edge length h. On each square we have a bilinear ansatz function

$$p(x, y) = \alpha + \beta x + \gamma y + \delta xy$$

with parameters $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. As numerical degree of freedoms for an unknown function, we use the function values at the vertices of the squares.

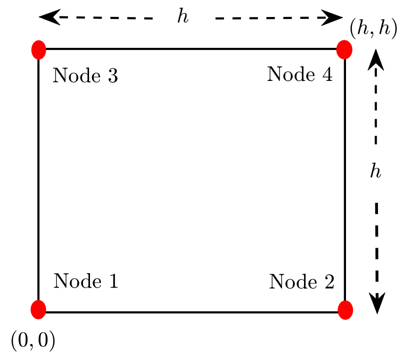

Abbildung 1: Square Q of edge length h with nodes labelled $1, 2, 3, 4$.

(a) Consider the square Q with vertices labelled $1, 2, 3$, and 4 as displayed in Figure 1. Show that for given point values at nodes $1, 2, 3, 4$ a unique bilinear function can be constructed. Do this by assuming point values $u_{1,2,3,4}$. Describe the bilinear function in one sentence or draw a simple(!) sketch. Can a unique bilinear function be constructed from three point values?

(b) On quadrilateral meshes, we typically consider the bilinear finite element space, i.e., the space of continuous piecewise bilinear functions, which is equipped with a basis of hat functions.
Let $\varphi_2(x, y)$ and $\varphi_3(x, y)$ denote the hat functions associated with the nodes 2 and 3 respectively and let $N_2(x, y)$ and $N_3(x, y)$ denote the element form functions on the square Q associated with the nodes 2 and 3 respectively. Reduce the generic bilinear function to explicit expressions for $N_2(x, y)$ and $N_3(x, y)$.

(c) Consider the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx$$

Compute the contribution of element Q to the matrix entry $A_{2,3} = a(\varphi_2, \varphi_3)$.

3 + 2 + 2 = 7 Points

Name: Matriculation-Nr.:

Aufgabe 5.

For $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ consider the following system of linear conservation laws

$$\partial_t U(t,x) + A\partial_x U(t,x) = 0,$$

with matrix $A = \begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix}$ and initial condition

$$U(0,x) = \begin{cases} \begin{pmatrix} 1 \\ 3 \end{pmatrix} & x < 0, \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} & x > 0. \end{cases}$$

Derive the solution by performing the following steps:

a) Diagonalize the system and transform the initial condition to reduce the problem into two independent scalar linear conservation laws by changing variables. For this purpose use the eigenvalue decomposition of the matrix $A = T\Lambda T^{-1}$ with a diagonal matrix Λ.

b) Solve the acquired independent scalar conservation laws with corresponding initial conditions.

c) Transform the solutions back to the variable $U(t,x)$.

3 + 1 + 3 = 7 Points

Name: Matriculation-Nr.:

Aufgabe 6.

We consider the *homogeneous shallow flow equations* given by

$$\partial_t U(t,x) + \partial_x F(U(t,x)) = 0.$$

The vector U and matrix $F(U)$ are given as

$$U := \begin{pmatrix} h \\ hu \end{pmatrix} \quad \text{and} \quad F(U) := \begin{pmatrix} hu \\ \alpha h u^2 + g h^2/2 \end{pmatrix},$$

with the gravitational acceleration $g \in \mathbb{R}^+$ and a modeling parameter $\alpha \in \mathbb{R}$ describing the velocity profile of the flow.
The above system can be written in quasi-linear form:

$$\partial_t U + A(U)\partial_x U = 0.$$

a) Compute the flux *Jacobian* $A(U) = DF(U)$.

b) What are the characteristic velocities of the system?

c) Under which condition is the system hyperbolic?

2 + 2 + 2 = 6 Points

Aufgabe 7.

Consider the linear transport equation

$$u_t + au_x = 0$$

with $a \in \mathbb{R}$. Using *Lax-Friedrichs* numerical method given by

$$u_j^{n+1} - \left(\frac{u_{j+1}^n + u_{j-1}^n}{2}\right) + \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) = 0,$$

you should:

a) Determine the *order of consistency* of the method.

b) Determine the modified equation. Use the modified equation to find a restriction on the grid parameters Δt and Δx such that stability holds.

c) Write Lax-Friedrichs method in conservation form and specify the numerical flux function.

d) Show that Lax-Friedrichs method is monotone.

2 + 3 + 1 + 3 = 9 Points

Name: Matriculation-Nr.:

Aufgabe 8.

The *super-bee limiter* for the nonlinear reconstruction of cell slopes in a finite volume method is given by

$$\phi^{(\mathrm{sb})}(\theta) = \max\left(0, \min\left(1, 2\theta\right), \min\left(2, \theta\right)\right)$$

(a) Draw the super-bee limiter in the θ-ϕ-diagram together with the minmod and van-Leer limiter.

(b) Is the superbee-limiter consistent and TVD?

(c) Consider the reconstruction formula $\widetilde{u}_i(x) = u_i + \sigma_i(x - x_i)$ with $\sigma_i = \phi(\theta_i)\frac{u_{i+1}-u_i}{\Delta x}$ and the values

1. $u_{i-1} = 3, \quad u_i = 4, \quad u_{i+1} = 10$ and
2. $u_{i-1} = 2, \quad u_i = 0, \quad u_{i+1} = 6.$

Compute the slope σ_i based on the super-bee and minmod limiter.

3 + 2 + 3 = 8 Points

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