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Applied and Computational Mathematics

RWTH AACHEN UNIVERSITY

Open the exam only after instruction by the assistants!

Partial Differential Equations (CES+SISC) | WS 2021
Exam | 09. March 2022

Allowed resources:

- Use only permanent-ink pens and no red-ink (or similar).
- Two hand-written, two-sided A4-papers with name and matriculation number.
- Other resources such as mobile phones, laptops etc. are not allowed.

Hint:

- Bringing resources which are specifically not allowed to possess at the seat in the exam is considered to be a cheating attempt.
- In total, you have **150 minutes** time to work on the exam.
  All answers need to be explained sufficiently.
- To pass the exam you need to have at least **50%** of the total points.
- The exam review takes place on 08. April 2022 starting at 10:15 – 12:45 in Room 201 in Hauptgebäude (1010|201), Templergraben 55, 52062 Aachen. Appointments for the oral repeat-exam have to be arranged at the exam review.
- Please answer the questions starting on the page where the questions are posed. If you need additional space you can use the empty pages reserved at the end of the exam sheets. In this case please write your name and matriculation number on the respective pages as well as the question number.
- With your signature you confirm in all conscience that you feel well enough to take the exam and that you will not attempt cheating.

Matriculation number: ___ ___ ___ ___ ___ ___

Last name, first name: ______________________

Signature: ______________________

Task	1	2	3	4	5	6	7	8	$\sum$
Points	6	8	9	7	8	7	7	8	60
Your points									

Exam + Bonus = Total

Grade:

Aufgabe 1.

For each of the following functions defined on Ω check whether it is in $H^1(\Omega)$ and $\mathcal{C}^1(\Omega)$. Justify your answer. If it belongs to $H^1(\Omega)$, calculate the weak derivatives for the functions that are in $H^1(\Omega)$.

a) $\Omega = \mathbb{R}$,

$$u_1(x) = \begin{cases} (x+2)^2, & -2 < x < 0, \\ (x-2)^2, & 0 \leq x < 2, \\ 0, & \text{else.} \end{cases}$$

b) $\Omega = (-1, 1)$,

$$u_2(x) = |x|.$$

c) $\Omega = \mathbb{R}$,

$$u_3(x) = \begin{cases} (x+1)^2, & x < 0, \\ (x-1)^2 + 1, & x \geq 0. \end{cases}.$$

2 + 2 + 2 = 6 Points

Aufgabe 2.

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain. Consider the following *Robin* boundary value problem. Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u + (1-\alpha)\, u &= f \quad \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} + \beta u &= 0 \quad \text{on } \partial\Omega, \end{cases} \qquad \text{where} \quad f \in L^2(\Omega).$$

a) Derive the weak formulation of the above problem.

b) Find the values of α and β for which a unique weak solution exists.

4 + 4 = 8 Points

Aufgabe 3.

Consider the boundary value problem

$$\begin{aligned} -\Delta u(x,y) &= 1 \quad \text{in } \Omega = (0,1)^2 \subset \mathbb{R}^2, \\ u(x,y) &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Choose the basis function as follows

$$\psi_{i,j}(x,y) = \sin(i\pi x)\sin(j\pi y), \quad 1 \le i,j \le N.$$

a) Derive the *Ritz-Galerkin* equations and find the stiffness matrix.

b) Choose four basis functions and find the approximate solution.

4 + 5 = 9 Points

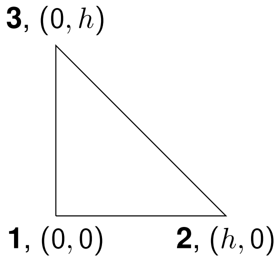
Name: Matriculation-Nr.:

Aufgabe 4.

For $\Omega \subset \mathbb{R}^2$, we consider a uniform triangular grid with each element of edge length h. On each triangle we have a quadratic ansatz function

$$p(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy,$$

with $\{a_i\}_{i=1}^6 \in \mathbb{R}$.



a) How many nodes are needed in addition to the triangle corners to guarantee a unique second order polynomial? Include your choice of nodes into the sketch and verify uniqueness.

b) Let $\varphi_1(x,y)$ and $\varphi_2(x,y)$ denote the quadratic hat functions associated with nodes 1 $(0,0)$ and 2 $(h,0)$, respectively, and let $N_1(x,y)$ and $N_2(x,y)$ denote the element form functions with the nodes 1 and 2, respectively. Find explicit expressions for $N_1(x,y)$ and $N_2(x,y)$.

4 + 3 = 7 Points

Name: Matriculation-Nr.:

Aufgabe 5.

We consider the *homogeneous shallow flow equations* given as follows

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\Big(hu^2 + \frac{1}{2}\left(gh^2\right)\Big) = 0, \end{cases}$$

with g being a positive parameter, so-called gravitational acceleration. We write the two equations as a system

$$\partial_t U + \partial_x F(U) = 0.$$

a) Define U, and $F(U)$.

b) We linearize the given system as the following expression

$$\partial_t U + A(U)\partial_x U = 0.$$

Compute the flux *Jacobian* $A(U) = DF(U)$.

c) What are the characteristic velocities of the system?

d) Under which condition is the system hyperbolic?

2 + 2 + 2 + 2 = 8 Points

Name: Matriculation-Nr.:

Aufgabe 6.

For $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ consider the following system of linear conservation laws

$$\partial_t U(x,t) + A\partial_x U(x,t) = 0,$$

with $A := \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ and the initial condition given as follows

$$U(x,0) = \begin{cases} \begin{pmatrix} 0 \\ 4 \end{pmatrix}, & x < 0, \\ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, & x > 0. \end{cases}$$

Derive the solution by performing the following steps:

a) Diagonalise the system and transform the initial condition to reduce it to two independent scalar linear conservation laws by changing variables. Regarding this purpose use the eigenvalue decomposition of the matrix A as

$$A = T\Lambda T^{-1},$$

with Λ being diagonal matrix of eigenvalues.

b) Solve the acquired independent scalar conservation laws with the use of the corresponding initial conditions.

c) Transform the solutions back to the variable $U(x, t)$.

3 + 1 + 3 = 7 Points

Aufgabe 7.

Consider the following advection equation

$$\partial_t u + a\partial_x u = 0, \qquad \forall\, a \in \mathbb{R},\, x \in \Omega.$$

We will discretize the above equation by using finite differences with the *Lax-Friedrichs* scheme in the following way

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) + \frac{a\lambda}{2}(u_{j-1}^n - u_{j+1}^n),$$

where u_j^n represents the solution at the j-th grid point and the n-th time step. The factor λ is the ratio between Δt and Δx, i.e. $\lambda := \Delta t/\Delta x$.

a) Show that the above scheme is consistent up to at least first order.

b) Write the update rule as time update $\mathcal{H}_{\Delta t}(u_{j-1}^n, u_j^n, u_{j+1}^n; j)$, compute the partial derivatives $\partial_* \mathcal{H}$ and use these quantities to assess, whether the scheme is *monotone*.

c) Write the scheme in *conservative form* by specifying the numerical flux functions.

3 + 2 + 2 = 7 Points

Aufgabe 8.

The so-called *Superbee* limiter (SB) for the nonlinear reconstruction of cell slopes in the finite volume method is given by

$$\phi^{(\mathsf{SB})}(\theta) = \max\Big(0, \min(1, 2\theta), \min(2, \theta)\Big).$$

a) Draw the *Superbee* limiter in the θ-ϕ-diagram together with *van Leer* limiter (VL) and the *minmod* limiter (MM).

b) Is the *Superbee* limiter consistent and TVD?

c) Consider the reconstruction formula $\widetilde{u}_j(x) = u_j + \sigma_j(x - x_j)$ with

$$\sigma_j = \phi(\theta_j)\frac{u_{j+1} - u_j}{\Delta x},$$

and the values

$$\begin{cases} u_{j-1} = 1, \\ u_j = 0, \\ u_{j+1} = 8. \end{cases}$$

Compute the slope σ_j based on the *Superbee* and *minmod* limiter.

3 + 2 + 3 = 8 Points

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Good luck!