Prof. Dr. Manuel Torrilhon

**Öffnen Sie den Klausurbogen erst nach Aufforderung!**

# Partial Differential Equations (CES+SISC) | WS 2019/20
# Klausur | 10. March 2020

**Zugelassene Hilfsmittel:**

- Dokumentenechtes Schreibgerät, aber kein Rotstift.
- Zwei eigenhändig und beidseitig beschriebene DIN A4 Blätter, die mit Namen und Matrikelnummer versehen sind.
- Weitere Hilfsmittel, insbesondere die Nutzung eines Taschenrechners, sind nicht erlaubt.

**Hinweise:**

- Das Mitführen von Mobilfunkgeräten während der Klausur gilt als Täuschungsversuch.
- Sie haben insgesamt **150 Minuten** Zeit zur Bearbeitung. *Alle Antworten sind ausführlich zu begründen.*
- Zum Bestehen der Klausur reichen **50%** der möglichen Punkte.
- Die Klausureinsicht findet am 17. March 2020 von 10:30–12:30 Uhr im (1090|334) klPhys, Rogowski Gebäude, Schinkelstraße 2 statt. Termine zur mündlichen Ergänzungsprüfung sind während der Klausureinsicht zu vereinbaren.
- Bitte beginnen Sie jede Aufgabe auf dem Blatt, auf dem die Aufgabenstellung formuliert ist. Sollten Sie außer der gegenüber befindlichen Leerseite noch eines der angehefteten Leerblätter benutzen, so geben Sie bitte auf dem ersten Blatt den Hinweis „Fortsetzung auf einem anderen Blatt“ an. *Bitte kennzeichnen Sie jedes Blatt mit Ihrem Namen und Ihrer Matrikelnummer – auch die benutzten Blanko-Blätter.*
- Durch Ihre Unterschrift versichern Sie, dass Sie zu Beginn der Klausur nach bestem Wissen prüfungsfähig sind und dass die Prüfungsleistung von Ihnen ohne nicht zugelassene Hilfsmittel erbracht wurde.

**Matrikelnummer:** ___ ___ ___ ___ ___ ___

**Name, Vorname:** ______________________________

**Unterschrift:** ______________________________

| Aufgabe | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\sum$ |
|---|---|---|---|---|---|---|---|---|---|
| Punkte | 7 | 7 | 8 | 8 | 7 | 7 | 7 | 9 | 60 |
| Ihre Punkte | | | | | | | | | |

Klausur ______ + Bonus ______ = Gesamt ______

Note: ______

**Aufgabe 1.**

Consider the following real-valued functions $u\colon \Omega \to \mathbb{R}$. Which of these are weakly differentiable? Justify your answer. Calculate the weak derivatives if they exist.

(a) $\Omega = (-1, 1)$ and $u\colon \Omega \to \mathbb{R}$ is given by

$$u(x) = 1 - |x|.$$

(b) $\Omega = (-1, 1)$ and $u\colon \Omega \to \mathbb{R}$ is given by

$$u(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 0 & -1 \leq x < 0. \end{cases}$$

(c) $\Omega = (0, 2)$ and $u\colon \Omega \to \mathbb{R}$ is given by

$$u(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 & 1 \leq x \leq 2. \end{cases}$$

**2+2+3 = 7 Punkte**

**Aufgabe 2.**

Let $\Omega \subset \mathbb{R}$ be an open bounded domain. Consider the following Robin boundary (which is a combination of Dirichlet and Neumann boundary conditions) value problem

$$\begin{aligned} -\Delta u + \alpha u &= f \quad \text{in } \Omega \\ \nabla u \cdot \mathbf{n} + \beta u &= g \quad \text{on } \partial\Omega \end{aligned}$$

where $u \in H^1(\Omega)$ and $f, g \in L^2(\Omega)$.

(a) Derive the weak formulation of the above problem.

(b) Find the values of $\alpha$ and $\beta$ for which there exists a unique solution.

**3+4 = 7 Punkte**

**Aufgabe 3.**

Consider the boudary value problem

$$\begin{aligned} -\Delta u(x,y) &= 1 \quad \text{in } \Omega = (0,1)^2 \subset \mathbb{R}^2 \\ u(x,y) &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Choose the basis functions as

$$\psi_{i,j}(x,y) = \sin(i\pi x)\sin(j\pi y), \quad 1 \le i,j \le N$$

(a) Derive the Ritz-Galerkin equations and find the stiffness matrix.

(b) Choose 4 basis functions and find the approximate solution.

**3+5 = 8 Punkte**

**Aufgabe 4.**

For $\Omega \subset \mathbb{R}^2$, we consider a uniform quadrilateral grid that is build from squares of equal edge length $h$. On each square we have a bilinear ansatz function

$$p(x, y) = \alpha + \beta x + \gamma y + \delta xy$$

with parameters $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. As numerical degree of freedoms for an unknown function, we use the function values at the vertices of the squares.

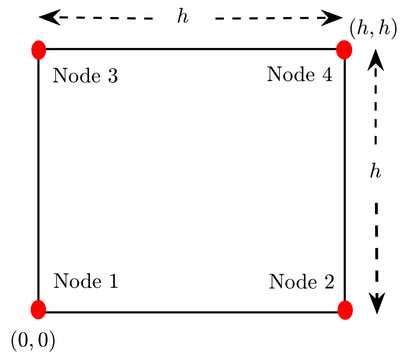


Abbildung 1: Square $Q$ of edge length $h$ with nodes labelled $1, 2, 3, 4$.

(a) Consider the square $Q$ with vertices labelled $1, 2, 3,$ and $4$ as displayed in Figure 1. Show that for given point values $u_{1,2,3,4}$, a unique bilinear function can be constructed.

(b) On quadrilateral meshes, we typically consider the bilinear finite element space, i.e., the space of continuous piecewise bilinear functions, which is equipped with a basis of bilinear hat functions.

Let $\varphi_1(x, y)$ and $\varphi_2(x, y)$ denote the bilinear hat functions associated with the nodes $1$ and $2$ respectively and let $N_1(x, y)$ and $N_2(x, y)$ denote the element form functions on the square $Q$ associated with the nodes $1$ and $2$ respectively. Find explicit expressions for $N_1(x, y)$ and $N_2(x, y)$.

(c) Consider the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx$$

Compute the contribution of element $Q$ to the matrix entry $A_{1,2} = a(\varphi_1, \varphi_2)$.

**3+2+3 = 8 Punkte**

**Aufgabe 5.**

For $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ we consider the Burgers equations

$$\partial_t u + \partial_x \left(\tfrac{1}{2}u^2\right) = 0$$

(a) Give the integral formulation of the equation for the domain $[a,b] \times [t_1,t_2] \subset \mathbb{R} \times \mathbb{R}^+$.

(b) What are the Rankine-Hugoniot conditions of the Burgers equation for a discontinuity? What is the shock speed?

(c) Formulate an entropy law for the Burgers equation with corresponding entropy $\eta(u)$ and entropy flux $h(u)$.

(d) The initial condition

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$$

allows a discontinuous shock solution and continuous, piece-wise differentiable rarefaction solution. Write down both solutions.

(e) Which solution is the correct entropy solution? (no further computations needed).

**1 + 1 + 1 + 2 + 2 = 7 Punkte**

**Aufgabe 6.**

For a following system of linear conservation laws

$$\partial_t U(t,x) + A\partial_x U(t,x) = 0,$$

with a matrix $A = \begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix}$ and an initial condition

$$U(0,x) = \begin{cases} \begin{pmatrix} 0 \\ 4 \end{pmatrix}, & x < 0, \\ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, & x > 0. \end{cases}$$

Consider following steps:

a) Diagonalise the system and transform the initial condition to reduce it to two independent scalar linear conservation laws by changing variables. For this purpose use the eigenvalue decomposition of the matrix $A = T\Lambda T^{-1}$ with a diagonal $\Lambda$.

b) Solve acquired independent scalar conservation laws with corresponding initial conditions.

c) Transform the solution back to the variable $U(t,x)$.

**3 + 1.5 + 2.5 = 7 Punkte**

**Aufgabe 7.**

Consider the following advection equation

$$\partial_t u + a\partial_x u = 0 \qquad \forall x \in \Omega.$$

We will discretize the above equation using finite differences and a Lax-Friedrichs scheme in the following way

$$u_i^{n+1} = \frac{1}{2}(u_{i-1}^n + u_{i+1}^n) + \frac{a\lambda}{2}(u_{i-1}^n - u_{i+1}^n).$$

where $u_i^n$ represents the solution at the $i$-th grid point and the $n$-th time step. The factor $\lambda$ is the ratio of $\Delta t$ and $\Delta x$ i.e. $\lambda = \Delta t/\Delta x$. Show that

a) The above LF scheme is consistent upto at least first order.

b) Let $g(\theta)$ be the amplification factor of the scheme, then

$$g(\theta) = \cos(\theta) - \mathrm{i}a\lambda \sin(\theta),$$

where $\theta = k\pi\Delta x$ with $k$ being the wave number of the Fourier ansatz.

c) For what ranges of $\lambda$ is the scheme stable?

**3 + 2 + 2 = 7 Punkte**

**Aufgabe 8.**

The *super-bee limiter* for the nonlinear reconstruction of cell slopes in a finite volume method is given by

$$\phi^{(\mathsf{sb})}(\theta) = \max\left(0, \min\left(1, 2\theta\right), \min\left(2, \theta\right)\right)$$

(a) Draw the super-bee limiter in the $\theta$-$\phi$-diagram together with the minmod limiter and the TVD region.

(b) Is the superbee-limiter consistent and TVD?

(c) Consider the reconstruction formulas

$$\widetilde{u}_i(x) = u_i + \sigma_i(x - x_i) \quad \text{with} \quad \sigma_i = \phi(\theta_i)\frac{u_{i+1} - u_i}{\Delta x}$$

and the values

$$u_{i-1} = 3, \quad u_i = 4, \quad u_{i+1} = 10,$$

and compute the slope $\sigma_i$ based on the super-bee and minmod limiter.

**3 + 2 + 4 = 9 Punkte**

Name: Matrikel-Nr.: