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Open the exam only after instruction by the assistants!

Partial Differential Equations (CES+SISC) | SS 2025
Exam | 15.09.2025

Allowed resources:

- Use only permanent pens with blue or black ink. Particularly **no** red ink or pencil is allowed.
- Two hand-written, two-sided A4-papers in the original with name and matriculation number. No printed out papers are allowed.
- Other resources such as mobile phones, laptops etc. are prohibited.

Hint:

- Bringing resources which are specifically not allowed to possess at the seat in the exam is considered to be a cheating attempt.
- In total, you have **150 minutes** time to work on the exam. *All answers need to be explained sufficiently*.
- To pass the exam you need to have at least **50%** of the total points.
- The exam review takes place on 29.09.2025 starting at 14:00 – 15:00 in klPhys (1090|334). Appointments for the oral repeat-exam have to be arranged at the exam review.
- Please answer the questions starting on the page where the questions are posed. If you need additional space you can use the empty pages reserved at the end of the exam sheets. In this case please write your name and matriculation number on the respective pages as well as the question number.
- With your signature you confirm in all conscience that you feel well enough to take the exam and that you will not attempt cheating.

Matriculation number: ___ ___ ___ ___ ___ ___

Last name, first name: ____________________

Signature: ____________________

Task	1	2	3	4	5	6	7	8	$\sum$
Points	5	9	8	8	6	8	8	8	60
Your points									

Exam [ ] + Bonus [ ] = Total [ ]

Grade: [ ]

Problem 1.

For each of the following functions defined on Ω check whether it is in $H^1(\Omega)$ and $\mathcal{C}^1(\Omega)$. Justify your answer. If it belongs to $H^1(\Omega)$, calculate the weak derivatives for the functions that are in $H^1(\Omega)$.

a) $\Omega = \mathbb{R}$,

$$u_1(x) = \begin{cases} (x+3)^2, & -3 < x < -1, \\ (x-1)^2, & -1 \leq x < 1, \\ 0, & \text{else.} \end{cases}$$

b) $\Omega = (-2, 0)$,

$$u_2(x) = |x+1|.$$

c) $\Omega = \mathbb{R}$,

$$u_3(x) = \begin{cases} (x+1)^2, & x < 0, \\ (x-1)^2 + 2, & x \geq 0. \end{cases}.$$

2 + 2 + 1 = 5 Points

Problem 2.

Let $\Omega \subset \mathbb{R}$ be an open bounded domain. Consider the following Robin boundary (which is a combination of Dirichlet and Neumann boundary conditions) value problem

$$\begin{aligned} -\frac{1}{\alpha}\Delta u + u &= f \quad \text{in } \Omega \\ \nabla u \cdot \boldsymbol{n} + \beta u &= g \quad \text{on } \partial\Omega \end{aligned}$$

where $f, g \in L^2(\Omega),\ \alpha > 1,\ \beta \geq 0$ and $\boldsymbol{n}$ is the outer unit normal vector at $\partial\Omega$.

a) Derive the weak formulation of the above problem, including the statement of reasonable function spaces for the solution and test functions.

b) Prove that for any $\alpha > 1$ and $\beta \geq 0$ there exists a unique solution.

3 + 6 = 9 Points

Name: Mat-Nr.:

Problem 3.

Let V be a Hilbert space and consider a bilinear form $a : V \times V \to \mathbb{R}$ as well as a linear form $b : V \to \mathbb{R}$ that satisfy the conditions of the Lax-Milgram theorem. Let also V_N be an N-dimensional subspace of V.

(a) Derive a linear system of equations for the Ritz-Galerkin solution $u_N \in V_N$ of the variational formulation in V defined by a and b.

(b) Which properties does the Ritz-Galerkin matrix A_N have? Is the linear Ritz-Galerkin system uniquely solvable?

(c) Consider the following minimization problem:

$$\min_{v \in V} J(v), \qquad \text{with } J(v) = \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} - \int_\Omega v \,\mathrm{d}x,$$

where $V = H^1_0(\Omega)$ and $\Omega = (0,\pi) \times (0,\pi) \subset \mathbb{R}^2$. What is the corresponding PDE to this minimization problem and weak formulation. Use the M-dimensional subspace V_M spanned by the following basis functions:

$$\psi_{i,j}(x,y) = \sin(ix)\sin(jy), \qquad 1 \le i,j \le N, \qquad M = N^2,$$

to derive elements of the corresponding Ritz-Galerkin stiffness matrix and right hand side.

Hint: You can use without a proof the following integrals:

$$\int_0^\pi \sin(kx) = \frac{-1+\cos(k\pi)}{k}, \quad \text{for } k \in \mathbb{N},$$

$$\int_\Omega \begin{pmatrix} i\cos(ix)\sin(jy) \\ j\cos(jy)\sin(ix) \end{pmatrix} \begin{pmatrix} k\cos(kx)\sin(ly) \\ l\cos(ly)\sin(kx) \end{pmatrix} \mathrm{d}\Omega = \begin{cases} \frac{(i^2+j^2)\pi^2}{4}, & (i,j) = (k,l) \\ 0, & \text{otherwise} \end{cases}.$$

3 + 1 + 4 = 8 Points

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Problem 4.

Consider the Poisson equation

$$\begin{aligned} \Delta u &= f \quad x \in \Omega \\ u &= 0 \quad x \in \partial\Omega \end{aligned}$$

on a 1-dimensional domain $\Omega = (-1, 1)$ with some smooth source function $f : \mathbb{R} \to \mathbb{R}$, i.e., $f \in C^\infty(\mathbb{R})$. For the solution we want to use the Finite-Element-Method with different basis functions on an equidistant grid $x_0, ...x_{n+1}$ with

$$x_k = -1 + kh,$$

where $h = 2/(n + 1)$ is the mesh width.

a) Provide an explicit expression for the P1-basis function $\varphi_k(x)$ on the inner nodes $x_k,\ k = 1, \dots, n$. What are the two element form funcions $N_j^{(1)}(\xi)\,(j = 1, 2)$ on a reference element $[0, 1]$?

b) Compute the system matrix for the Poisson problem with P1-elements by first computing the element system matrix and then combining.

c) For P2-elements we add additional nodes at positions $x_{k+\frac{1}{2}}$ in the middle of each element. What are the element form functions $N_j^{(2)}(\xi)\,(j = 1, 2, 3)$, associated with the local nodes $\xi \in \{0, \frac{1}{2}, 1\}$?

d) Compute the entries (2,2) and (1,3) of the 3×3 element system matrix.

e) What is the global matrix bandwidth (, i.e., the smallest integer K for which $A(i, j) = 0$ for all i, j with $|i - j| > K$) for P1- and P2-elements?

1.5 + 2 + 1.5 + 2 + 1 = 8 Points

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Problem 5.

We consider the *homogeneous shallow flow equations* given as follows

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\Big(hu^2 + \frac{1}{2}\left(gh^2\right)\Big) = 0, \end{cases}$$

with g being a positive parameter, so-called gravitational acceleration. We write the two equations as a system

$$\partial_t U + \partial_x F(U) = 0.$$

a) Define U, and $F(U)$.

b) We linearize the given system as the following expression

$$\partial_t U + A(U)\partial_x U = 0.$$

Compute the flux *Jacobian* $A(U) = DF(U)$.

c) What are the characteristic velocities of the system?

d) Under which condition is the system hyperbolic?

2 + 4 = 6 Points

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Problem 6.

For a following system of linear conservation laws

$$\partial_t U(t,x) + A\partial_x U(t,x) = 0,$$

with a matrix $A = \begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix}$ and an initial condition

$$U(0,x) = \begin{cases} \begin{pmatrix} 0 \\ 4 \end{pmatrix}, & x < 0, \\ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, & x > 0. \end{cases}$$

Consider following steps:

a) Diagonalise the system and transform the initial condition to reduce it to two independent scalar linear conservation laws by changing variables. For this purpose use the eigenvalue decomposition of the matrix $A = T\Lambda T^{-1}$ with a diagonal Λ.

b) Solve acquired independent scalar conservation laws with corresponding initial conditions.

c) Transform the solution back to the variable $U(t,x)$.

4 + 1 + 3 = 8 Points

Problem 7.

For the advection equation

$$\partial_t u + a\partial_x u = 0 \tag{$\star$}$$

with periodic boundaries we consider an equidistant discretization with time step Δt and grid size Δx, and the time-step-method

$$u_j^{n+1} = u_j^n + \frac{\nu}{2}\left(u_{j-1}^n - u_{j+1}^n\right) + \frac{\delta}{2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right) \tag{$\star\star$}$$

for point values $u_j^n \approx u(x_j, t_n)$. The Courant number is given by $\nu = \frac{a\Delta t}{\Delta x}$ and there is an additional parameter $\delta \in \mathbb{R}$.

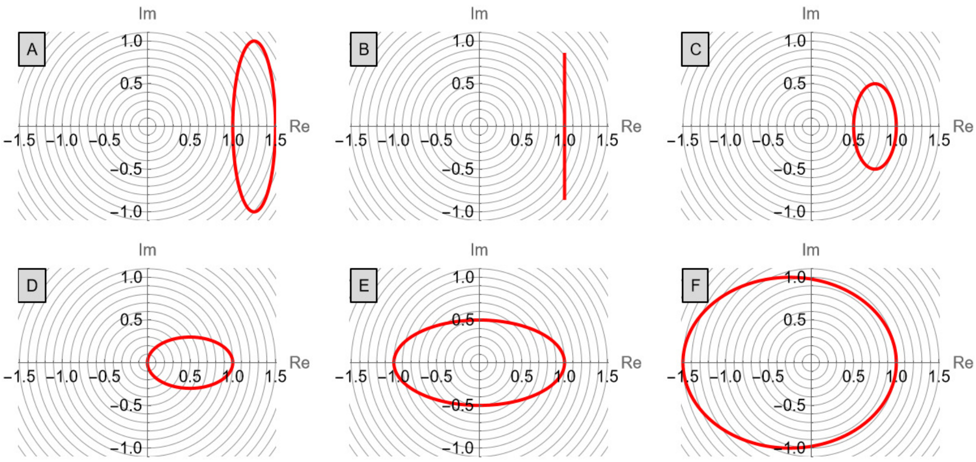
(a) For general δ, compute the local error and consistency order for the scheme ($\star\star$). What value of δ is needed to make the scheme second-order?

(b) For general δ show that amplification function of the method is given by

$$g(\xi) = 1 + \delta\left(\cos\xi - 1\right) - \mathbf{i}\,\nu\sin\xi$$

based on the Fourier or von-Neumann analysis.

(c) Consider the following 6 parametric plots of the amplification function $g(\xi)$ in the complex plane.



What are the values of the parameters ν and δ for each of these plots? In what cases do you expect a stable method in the sense of von-Neumann? Please justify!

(d) Name the methods for $\delta = \nu$ and $\delta = \nu^2$.

3 + 2 + 2 + 1 = 8 Points

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Problem 8.

Consider the following numerical schemes for the linear advection equation

$$u_t + au_x = 0, \quad \text{with} \quad a > 0.$$

Rewrite the scheme in the incremental form as follows

$$u_i^{n+1} = u_i^n + C_{i+1/2}^n \left(u_{i+1}^n - u_i^n\right) - D_{i-1/2}^n \left(u_i^n - u_{i-1}^n\right).$$

Compute the corresponding $D_{i-1/2}^n$ and $C_{i+1/2}^n$, and comment on the TVD property.

a) The Upwind scheme with the correct left hand-side stencil (as $a > 0$):

$$u_i^{n+1} = u_i^n - c\left(u_i^n - u_{i-1}^n\right).$$

b) The Upwind scheme with the right-hand side stencil:

$$u_i^{n+1} = u_i^n - c\left(u_{i+1}^n - u_i^n\right).$$

c) The *Lax-Friedrich* scheme:

$$u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{c}{2}\left(u_{i+1}^n - u_{i-1}^n\right).$$

Note in passing that $c = \dfrac{a\Delta t}{\Delta x}$ is the CFL number.

3 + 2 + 3 = 8 Points

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