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Applied and Computational Mathematics | RWTH AACHEN UNIVERSITY

Open the exam only after instruction by the assistants!

Partial Differential Equations (CES+SISC) | SS 2024
Exam | 19.09.2024

Allowed resources:

- Use only permanent-ink pens and no red-ink (or similar).
- Two hand-written, two-sided A4-papers with name and matriculation number.
- Other resources such as mobile phones, laptops etc. are not allowed.

Hint:

- Bringing resources which are specifically not allowed to possess at the seat in the exam is considered to be a cheating attempt.
- In total, you have **150 minutes** time to work on the exam.
  All answers need to be explained sufficiently.
- To pass the exam you need to have at least **60%** of the total points.
- The exam review takes place on 04.10.2024 starting at 15:00 – 16:00 in ACoM seminar room (1090|328). Appointments for the oral repeat-exam have to be arranged at the exam review.
- Please answer the questions starting on the page where the questions are posed. If you need additional space you can use the empty pages reserved at the end of the exam sheets. In this case please write your name and matriculation number on the respective pages as well as the question number.
- With your signature you confirm in all conscience that you feel well enough to take the exam and that you will not attempt cheating.

Matriculation number: ___ ___ ___ ___ ___ ___

Last name, first name: ______________________

Signature: ______________________

Task	1	2	3	4	5	6	7	8	$\sum$
Points	7	7	7	9	7	9	6	8	60
Your points									

Exam		Bonus		Total
	+		=	

Grade:

Aufgabe 1.

We consider periodic, real-valued functions u on the intervall $[-1, 1] \subset \mathbb{R}$. They can be written as Fourier series

$$u(x) = \sum_{k \in \mathbb{Z}} \alpha_k e^{\mathbf{i}k\pi x} \qquad x \in [-1, 1]$$

with Fourier coefficients $\alpha_k \in \mathbb{C}$. The integral satisfies

$$\int_{-1}^{1} |u(x)|^2 \, dx = \sum_{k \in \mathbb{Z}} |\alpha_k|^2$$

(a) First, consider the H^1-function $u : [a, b] \to \mathbb{R}$. What is the definition of the Sobolev-norm $\|u\|_{H^1([a,b])}$?

(b) Show that for periodic, real-valued functions u on $[-1, 1]$ we have the relation

$$\|u\|_{H^1([-1,1])} = \sqrt{\sum_{k \in \mathbb{Z}} (1 + k^2\pi^2) |\alpha_k|^2}$$

for the Sobolev-norm.

(c) Show that the one-dimensional Poisson problem

$$-\Delta u = f \qquad x \in [-1, 1]$$

for a periodic function f has the periodic solution

$$u(x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\beta_k}{k^2 \pi^2} e^{\mathbf{i}k\pi x}$$

where β_k, $k \in \mathbb{Z}$ are the Fourier coefficients of the given function $f(x)$.

Hint: You can compare the Fourier coefficients on both sides of the equation due to orthogonality.

Remark: We ignored the coefficient at $k = 0$ assuming a zero mean of u.

(d) Compute the H^1-norm of the Poisson solution $u(x)$ from (c) in terms of its Fourier coefficients and proof the relation

$$\|u\|_{H^1([-1,1])} \leq \|f\|_{L^2([-1,1])}$$

for periodic solutions.

1 + 2 + 2 + 2 = 7 Points

Aufgabe 2.

Let $\Omega \subset \mathbb{R}^2$ be open and bounded. The boundary value problem is considered

$$\begin{aligned} -\Delta u + \alpha u &= f \quad \text{in } \Omega, \\ \nabla u \cdot \boldsymbol{n} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with $u \in H^1(\Omega)$ and $f \in L^2(\Omega)$, $\alpha \in \mathbb{R}$ and $\alpha > 0$.

(a) Derive the weak formulation of the above problem.

(b) Derive the bilinear form resulting from the boundary value problem. Show that it is coercive in the $H^1(\Omega)$-norm.

(c) Is the linear form describing the right-hand side continuous? Justify your answer.

3 + 2 + 2 = 7 Points

Name: Mat-Nr.:

Aufgabe 3.

(a) Consider the boundary value problem

$$-\Delta u(x,y) = 1 \quad \text{in } \Omega = (0,1)^2 \subset \mathbb{R}^2$$
$$u(x,y) = 0 \quad \text{on } \partial\Omega$$

with general basis functions $\psi_{i,j}(x,y)$, $i,j = 1 \leq i,j \leq N$ which fulfill the boundary conditions. Derive the Ritz-Galerkin equations and write down the general integral expressions for the stiffness matrix and the right-hand side.

(b) Now consider the corresponding 1D boundary value problem

$$-u''(x) = 1 \quad \text{in } \Omega = (0,1) \subset \mathbb{R}$$
$$u(0) = u(1) = 0$$

(i) Find proper, **polynomial** basis functions $\psi_i(x)$ which satisfy the boundary conditions.

(ii) Find an analytical expression for the entries of the stiffness matrix of the corresponding Ritz-Galerkin equations.

Hint: You can use the derivation of the stiffness matrix from (a) to solve (b)(ii).

3 + (1 + 3) = 7 Points

Aufgabe 4.

For $\Omega \subset \mathbb{R}^2$, we consider a uniform quadrilateral grid that is build from squares of equal edge length h. On each square we have a bilinear ansatz function

$$p(x, y) = \alpha + \beta x + \gamma y + \delta xy$$

with parameters $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. As numerical degree of freedoms for an unknown function, we use the function values at the vertices of the squares.

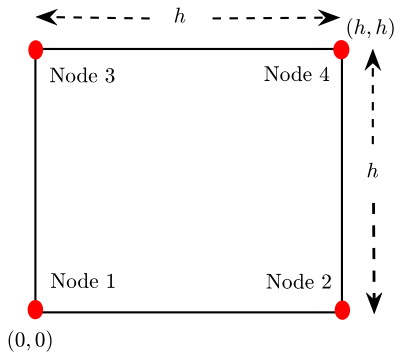

Abbildung 1: Square Q of edge length h with nodes labelled $1, 2, 3, 4$.

(a) Consider the square Q with vertices labelled $1, 2, 3,$ and 4 as displayed in Figure 1. Show that for given point values $u_{1,2,3,4}$, a unique bilinear function can be constructed.

(b) On quadrilateral meshes, we typically consider the bilinear finite element space, i.e., the space of continuous piecewise bilinear functions, which is equipped with a basis of bilinear hat functions.

Let $\varphi_1(x, y)$ and $\varphi_2(x, y)$ denote the bilinear hat functions associated with the nodes 1 and 2 respectively and let $N_1(x, y)$ and $N_2(x, y)$ denote the element form functions on the square Q associated with the nodes 1 and 2 respectively. Find explicit expressions for $N_1(x, y)$ and $N_2(x, y)$.

(c) Consider the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx$$

Compute the contribution of element Q to the matrix entry $A_{1,2} = a(\varphi_1, \varphi_2)$.

3 + 3 + 3 = 9 Points

Name: Mat-Nr.:

Aufgabe 5.

We consider the *homogeneous shallow flow equations* given as follows

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\Big(hu^2 + \frac{1}{2}\left(gh^2\right)\Big) = 0, \end{cases}$$

with g being a positive parameter, so-called gravitational acceleration. We write the two equations as a system

$$\partial_t U + \partial_x F(U) = 0.$$

a) Define U, and $F(U)$.

b) We linearize the given system as the following expression

$$\partial_t U + A(U)\partial_x U = 0.$$

Compute the flux *Jacobian* $A(U) = DF(U)$.

c) What are the characteristic velocities of the system?

d) Under which condition is the system hyperbolic?

1 + 2 + 2 + 2 = 7 Points

Name: Mat-Nr.:

Aufgabe 6.

Consider the linear system of conservation laws

$$\partial_t U + A\partial_x U = 0 \tag{\#}$$

with matrix $A = \begin{pmatrix} 1 & 1 \\ \varepsilon^2 & 1 \end{pmatrix}$ using $\varepsilon > 0$ for a solution vector $U : \mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}^2$.

(a) Show that the system is hyperbolic for all $\varepsilon > 0$. What happens to hyperbolicity in the special case $\varepsilon = 0$?

(b) Show that $v_{1,2} = (\pm\frac{1}{\varepsilon}, 1)^T$ are the eigenvectors of the matrix A. What are the eigenvalues?

(c) Diagonalize the hyperbolic system $(\#)$ for general $\varepsilon > 0$.

(d) Consider the initial conditions

$$U(x,0) = \begin{cases} \begin{pmatrix} 2 \\ 2 \end{pmatrix}, & x < 0 \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & x > 0 \end{cases}$$

for the system $(\#)$. What is the resulting intermediate state $U^\star$ of the corresponding Riemann solution for general $\varepsilon > 0$.

(e) Give a sketch of the Riemann solution for small ε. Why is the solution called *"delta-shock"* in the limit $\varepsilon \to 0$?

2 + 2 + 1 + 2 + 2 = 9 Points

Name: Mat-Nr.:

Aufgabe 7.

Consider the following initial value problem

$$u_t + u_x = 0 \qquad \text{for } t > 0, x \in \mathbb{R},$$
$$u(0,x) = u_0(x) \qquad \text{for } x \in \mathbb{R}.$$

For its solution the following numerical scheme has been suggested

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(u_{j+1}^n - u_j^n\right),$$

with

$$u_j^n \approx \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u\left(t^n, x\right)\, dx,$$

on an equidistant grid with $t^n = n\Delta t$ and $x_j = j\Delta x$ where $n \in \mathbb{N}_0$ and $j \in \mathbb{Z}$.

a) Asses whether or not the scheme is conservative.

b) Determine the consistency order of the scheme.

c) Asses whether or not the scheme is monotone.

2 + 2 + 2 = 6 Points

Name: Mat-Nr.:

Aufgabe 8.

The *super-bee limiter* for the nonlinear reconstruction of cell slopes in a finite volume method is given by

$$\phi^{(\mathrm{sb})}(\theta) = \max\left(0, \min\left(1, 2\theta\right), \min\left(2, \theta\right)\right)$$

(a) Draw the super-bee limiter in the θ-ϕ-diagram together with the minmod and van-Leer limiter.

(b) Is the superbee-limiter consistent and TVD?

(c) Consider the reconstruction formula $\widetilde{u}_i(x) = u_i + \sigma_i(x - x_i)$ with $\sigma_i = \phi(\theta_i)\frac{u_{i+1}-u_i}{\Delta x}$ and the values

1. $u_{i-1} = 3, \quad u_i = 4, \quad u_{i+1} = 10$ and
2. $u_{i-1} = 2, \quad u_i = 0, \quad u_{i+1} = 6.$

Compute the slope σ_i based on the super-bee and minmod limiter.

3 + 2 + 3 = 8 Points

Name: Mat-Nr.: