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Applied and Computational Mathematics | RWTH AACHEN UNIVERSITY

Open the exam only after instruction by the assistants!

Partial Differential Equations (CES+SISC) | SS 2023
Exam | 18.09.2023

Allowed resources:

- Use only permanent-ink pens and no red-ink (or similar).
- Two hand-written, two-sided A4-papers with name and matriculation number.
- Other resources such as mobile phones, laptops etc. are not allowed.

Hint:

- Bringing resources which are specifically not allowed to possess at the seat in the exam is considered to be a cheating attempt.
- In total, you have **150 minutes** time to work on the exam.
  All answers need to be explained sufficiently.
- To pass the exam you need to have at least **50%** of the total points.
- The exam review takes place on 22.09.2023 starting at 12:00 – 14:00 in Kleine Physik (klPhys) (1090|334). Appointments for the oral repeat-exam have to be arranged at the exam review.
- Please answer the questions starting on the page where the questions are posed. If you need additional space you can use the empty pages reserved at the end of the exam sheets. In this case please write your name and matriculation number on the respective pages as well as the question number.
- With your signature you confirm in all conscience that you feel well enough to take the exam and that you will not attempt cheating.

Matriculation number: ___ ___ ___ ___ ___ ___

Last name, first name: ______________________

Signature: ______________________

Task	1	2	3	4	5	6	7	8	$\sum$
Points	6	9	8	7	8	8	6	8	60
Your points									

Exam [ ] + Bonus [ ] = Total [ ]

Grade: [ ]

Aufgabe 1.

For each of the following three functions defined on Ω, check whether they are in $H^1(\Omega)$. If yes, compute the weak derivative. Also discuss whether they are in $C^1(\Omega)$:

- $u_1(x) = \begin{cases} \sin x + \cos x & x \in (0, 2\pi) \\ 1 & \text{otherwise} \end{cases}, \quad \Omega = \mathbb{R}$
- $u_2(x) = \begin{cases} \cos x & x < 0 \\ 0 & \text{otherwise} \end{cases}, \quad \Omega = \mathbb{R}$
- $u_3(x) = \begin{cases} (x+2)^2 & -2 < x < 0 \\ (x-2)^2 & 0 \le x < 2 \\ 0 & \text{otherwise} \end{cases}, \quad \Omega = \mathbb{R}.$

2 + 2 + 2 = 6 Points

Aufgabe 2.

Consider the following problem with a Neumann boundary condition:

$$\begin{cases} \alpha\Delta u - u = -\alpha f(x) & \text{in } \Omega := (0,1)\times(0,1) \\ \partial_n u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f(x) = -x_1^2$ if $x = (x_1, x_2)$ and $\alpha > 0$.

(a) Derive the weak formulation of the above problem.

(b) Show existence and uniqueness of the weak solution.

(c) Comment on the existence of a weak solution for $\alpha \to \infty$.

3 + 4 + 2 = 9 Points

Name: Mat-Nr.:

Aufgabe 3.

Consider the boundary value problem

$$\frac{d^4u}{dx^4} = f \qquad a < x < b$$
$$u(a) = u'(a) = 0$$
$$u(b) = u'(b) = 0$$

(a) Show that this problem satisfies the variational formulation

$$\int_a^b u''(x)v''(x)dx = \int_a^b f(x)v(x)dx \quad \forall v \in W$$

with $W = \{u \in H^2(a,b),\ u(a) = u'(a) = u(b) = u'(b) = 0\}$.

(b) On the reference interval $[0,1] \subset \mathbb{R}$ consider the polynomial space

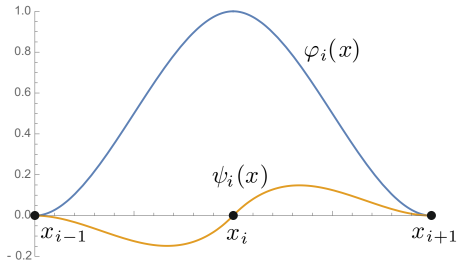
$$P_3 = \{p : \xi \mapsto p(\xi) \quad \text{polynomial}, \quad \deg(p) \leq 3\}.$$

What is the dimension of this space? Give a basis of this space by constructing form functions $\hat{N}_i(\xi)$, $i = 1,2,3,4$ such that each form function gives one for exactly one of the values $\{N_i(0),\ N_i(1),\ N_i'(0),\ N_i'(1)\}$ and zero for the other three.

(c) For an interval $\Omega = [a,b]$ we consider the positions $\{x_i\}_{i=0,\dots N+1} \subset [a,b]$ with $a = x_0 < x_i < x_{N+1} = b$. Lets define the intervals $K_i = [x_i, x_{i+1}]$ as elements and define the differentiable finite-element space

$$\hat{S}^3\left([a,b], \{K_i\}_{i=1}\right) = \left\{u \in C^1([a,b]),\ \ u(x)|_K = \text{polynomial with degree } 3\right\}.$$

As basis for this space we use hat functions φ_j such that at the points $\{x_i\}_{i=0,\dots N+1}$ we have $\varphi_j(x_i) = \delta_{ij}$ and $\varphi_j'(x_i) = 0$ for the derivatives, **as well as** shape functions ψ_j such that $\psi_j'(x_i) = \delta_{ij}$ and $\psi_j(x_i) = 0$



What is the dimension of $\hat{S}^3$ for $N+1$ elements?

(d) Let us choose $a = -1$ and $b = 1$ and consider the above boundary value problem with a discretization of the intervall $[a,b]$ by two elements $K_0 = [-1,0]$ and $K_1 = [0,1]$ each equipped with the space P_3. Using the basis from (c) and after exploiting the boundary values what are the two remaining basis functions of this discretization?

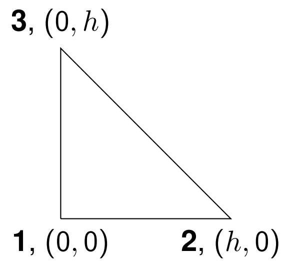
2 + 2 + 2 + 2 = 8 Points

Aufgabe 4.

For $\Omega \subset \mathbb{R}^2$, we consider a uniform triangular grid with each element of edge length h. On each triangle we have a quadratic ansatz function

$$p(x, y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy,$$

with $\{a_i\}_{i=1}^{6} \in \mathbb{R}$.



a) How many nodes are needed in addition to the triangle corners to guarantee a unique second order polynomial? Include your choice of nodes into the sketch and verify uniqueness.

b) Let $\varphi_1(x, y)$ and $\varphi_2(x, y)$ denote the quadratic hat functions associated with nodes 1 $(0, 0)$ and 2 $(h, 0)$, respectively, and let $N_1(x, y)$ and $N_2(x, y)$ denote the element form functions with the nodes 1 and 2, respectively. Find explicit expressions for $N_1(x, y)$ and $N_2(x, y)$.

4 + 3 = 7 Points

Name: Mat-Nr.:

Aufgabe 5.

For $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ consider the following system of linear conservation laws

$$\partial_t U(t,x) + A\partial_x U(t,x) = 0,$$

with matrix $A = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$ and initial condition

$$U(0,x) = \begin{cases} \begin{pmatrix} 1 \\ 3 \end{pmatrix} & x < 0, \\ \begin{pmatrix} -1 \\ 2 \end{pmatrix} & x > 0. \end{cases}$$

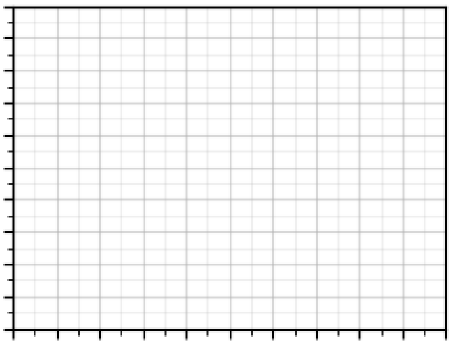
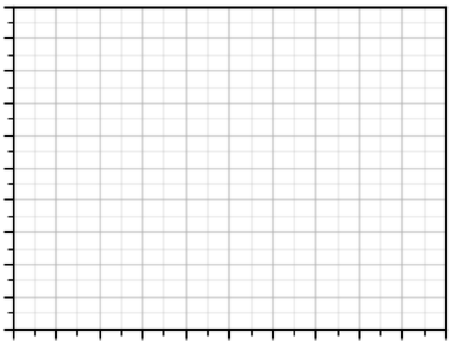
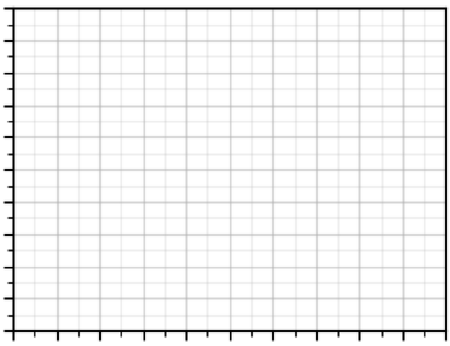
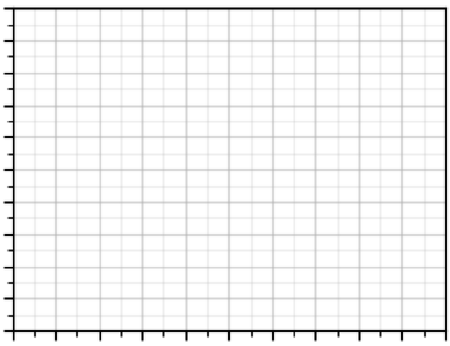
Derive the solution by performing the following steps:

a) Diagonalize the system and transform the initial condition to reduce the problem into two independent scalar linear conservation laws by changing variables. For this purpose use the eigenvalue decomposition of the matrix $A = T\Lambda T^{-1}$ with a diagonal matrix Λ.

b) Solve the acquired independent scalar conservation laws with corresponding initial conditions.

c) Transform the solutions back to the variable $U(t,x)$ and simplify the solution as much as possible.

d) Make a sketch of the solution for both, original and transformed variables, at time $t = 0$ and $t = 0.5$.

3 + 1 + 2 + 2 = 8 Points

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Aufgabe 6.

We consider the homogeneous shallow flow equations

$$\partial_t h + \partial_x (h\,u) = 0$$
$$\partial_t (h\,u) + \partial_x (h\,u^2 + \frac{1}{2} g\,h^2) = 0$$

with $g > 0$.
We can write the equations as a system

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{0}.$$

a) Define $\mathbf{U}$, define $\mathbf{F}(\mathbf{U})$.

b) We are now interessted in shocks. The state on the left of the shock will be denoted by $\mathbf{q}_l = (h_l, h_l\,u_l)$ and the right hand side of the shock by $\mathbf{q}_r = (h_r, h_r\,u_r)$. Write down the general Rankine-Hugoniot, and substitute in the shallow water system.

c) Determine if the following left and right states can be connected by single shock wave. If they can, compute the shock speed. Assume that $g = 10$.

1. $\mathbf{q}_l = (5, 25)$, $\mathbf{q}_r = (4, 26)$
2. $\mathbf{q}_l = (5, 25)$, $\mathbf{q}_r = (4, 24)$

d) Now consider the quasilinear version of the homogeneous shallow water equations, given by

$$\partial_t \mathbf{U} + A(\mathbf{U}) \partial_x \mathbf{U} = \mathbf{0}$$

with

$$A(\mathbf{U}) = \begin{pmatrix} 0 & 1 \\ -u^2 + g\,h & 2u \end{pmatrix}$$

Assume that we linearize the shallow water equatinos around state $\mathbf{q}_l$. Determine if the left and right states given in c) can be connected by a single shock wave. If they can, compute the shock speed. Assume that $g = 10$.

2 + 2 + 2 + 2 = 8 Points

Aufgabe 7.

Consider the following advection equation

$$\partial_t u + a\partial_x u = 0, \qquad \forall\, a \in \mathbb{R},\, x \in \Omega.$$

We will discretize the above equation by using finite differences with the *Lax-Friedrichs* scheme in the following way

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) + \frac{a\lambda}{2}(u_{j-1}^n - u_{j+1}^n),$$

where u_j^n represents the solution at the j-th grid point and the n-th time step. The factor λ is the ratio between Δt and Δx, i.e. $\lambda := \Delta t/\Delta x$.

a) Show that the above scheme is consistent up to at least first order.

b) Write the update rule as time update $\mathcal{H}_{\Delta t}(u_{j-1}^n, u_j^n, u_{j+1}^n; j)$, compute the partial derivatives $\partial_* \mathcal{H}$ and use these quantities to assess, whether the scheme is *monotone*.

c) Write the scheme in *conservative form* by specifying the numerical flux functions.

2 + 3 + 1 = 6 Points

Name: Mat-Nr.:

Aufgabe 8.

The *superbee* limiter for the nonlinear reconstruction of cell slopes in a finite volume method is given by

$$\phi^{(\mathrm{sb})}(\theta) = \max\left(0, \min\left(1, 2\theta\right), \min\left(2, \theta\right)\right)$$

(a) Draw the *superbee* limiter in the θ-ϕ-diagram together with the *minmod* and *van-Leer* limiter.

(b) Is the *superbee-limiter* consistent and TVD?

(c) Consider the reconstruction formula $\widetilde{u}_i(x) = u_i + \sigma_i(x - x_i)$ with $\sigma_i = \phi(\theta_i)\frac{u_{i+1}-u_i}{\Delta x}$ and the values

1. $u_{i-1} = 4, \quad u_i = 8, \quad u_{i+1} = 9$ and
2. $u_{i-1} = 4, \quad u_i = 9, \quad u_{i+1} = 8$.

Compute the reconstruction $\widetilde{u}_i(x_{i+1/2})$ based on the *superbee* and *minmod* limiter with $\Delta x = 1$.

3 + 2 + 3 = 8 Points

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