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Applied and Computational Mathematics | RWTH AACHEN UNIVERSITY

Open the exam only after instruction by the assistants!

Partial Differential Equations (CES+SISC) | SS 2022
Exam | 16.09.2022

Allowed resources:

- Use only permanent-ink pens and no red-ink (or similar).
- Two hand-written, two-sided A4-papers with name and matriculation number.
- Other resources such as mobile phones, laptops etc. are not allowed.

Hint:

- Bringing resources which are specifically not allowed to possess at the seat in the exam is considered to be a cheating attempt.
- In total, you have **150 minutes** time to work on the exam.
  All answers need to be explained sufficiently.
- To pass the exam you need to have at least **50%** of the total points.
- The exam review takes place on 10.10.2022 starting at 10:00 – 11:00 in Eph (1090|321). Appointments for the oral repeat-exam have to be arranged at the exam review.
- Please answer the questions starting on the page where the questions are posed. If you need additional space you can use the empty pages reserved at the end of the exam sheets. In this case please write your name and matriculation number on the respective pages as well as the question number.
- With your signature you confirm in all conscience that you feel well enough to take the exam and that you will not attempt cheating.

Matriculation number: ___ ___ ___ ___ ___ ___

Last name, first name: ______________________________

Signature: ______________________________

Task	1	2	3	4	5	6	7	8	$\sum$
Points	6	8	10	8	8	7	7	6	60
Your points									

Exam + Bonus = Total

Grade:

Aufgabe 1.

For each of the following functions defined on Ω check whether it is in $H^1(\Omega)$ and $\mathcal{C}^1(\Omega)$. Justify your answer. If it belongs to $H^1(\Omega)$, calculate the weak derivatives for the functions that are in $H^1(\Omega)$.

a) $\Omega = \mathbb{R}$,

$$u_1(x) = \begin{cases} \cos(x), & x \geq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

b) $\Omega = [-1, 1]$,

$$u_2(x) = \sqrt{|x|}$$

c) $\Omega = \mathbb{R}$,

$$u_3(x) = \begin{cases} -\pi, & x < 0, \\ \pi, & x \geq 0. \end{cases}$$

2 + 2 + 2 = 6 Points

Name: Matriculation-Nr.:

Aufgabe 2.

Let $\Omega = (0,1)^2 \subset \mathbb{R}^2$. Consider the following *Robin* boundary value problem. Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u + (1-\alpha)\, u &= f \quad \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} + \beta u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $f(x) = x_2^4$ if $x = (x_1, x_2)$.

a) Derive the weak formulation of the above problem.

b) Find the values of α and β for which a unique weak solution exists.

4 + 4 = 8 Points

Name: Matriculation-Nr.:

Aufgabe 3.

Let V be a Hilbert space and consider a bilinear form $a : V \times V \to \mathbb{R}$ as well as a linear form $b : V \to \mathbb{R}$ that satisfy the conditions of the Lax-Milgram theorem. Let also V_N be an N-dimensional subspace of V.

(a) Derive a linear system of equations for the Ritz-Galerkin solution $u_N \in V_N$ of the variational formulation in V defined by a and b.

(b) Which properties does the Ritz-Galerkin matrix A_N have? Is the linear Ritz-Galerkin system uniquely solvable?

(c) Consider the following minimization problem:

$$\min_{v \in V} J(v), \qquad J(v) = \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} - \int_\Omega v dx,$$

where $V = H^1_0(\Omega)$ and $\Omega = (0,\pi) \times (0,\pi) \subset \mathbb{R}^2$. Use the M-dimensional subspace V_M spanned by the following basis functions:

$$\psi_{i,j}(x,y) = \sin(ix)\sin(jy), \qquad 1 \leq i,j \leq N, \qquad M = N^2,$$

to derive elements of the corresponding Ritz-Galerkin stiffness matrix and right hand side.

4 + 2 + 4 = 10 Points

Name: Matriculation-Nr.:

Aufgabe 4.

For $\Omega \subset \mathbb{R}^2$, we consider a uniform quadrilateral grid that is build from squares of equal edge length h. On each square we have a bilinear ansatz function

$$p(x, y) = \alpha + \beta x + \gamma y + \delta xy$$

with parameters $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. As numerical degree of freedoms for an unknown function, we use the function values at the vertices of the squares.

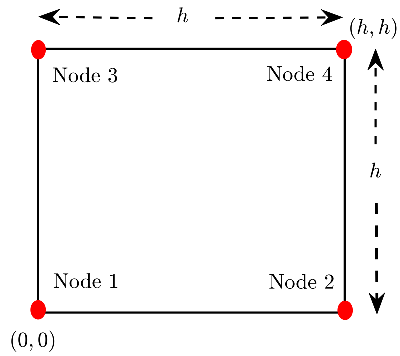

Abbildung 1: Square Q of edge length h with nodes labelled $1, 2, 3, 4$.

(a) Consider the square Q with vertices labelled $1, 2, 3,$ and 4 as displayed in Figure 1. Show that for given point values $u_{1,2,3,4}$, a unique bilinear function can be constructed.

(b) On quadrilateral meshes, we typically consider the bilinear finite element space, i.e., the space of continuous piecewise bilinear functions, which is equipped with a basis of bilinear hat functions.

Let $\varphi_1(x, y)$ and $\varphi_2(x, y)$ denote the bilinear hat functions associated with the nodes 1 and 2 respectively and let $N_1(x, y)$ and $N_2(x, y)$ denote the element form functions on the square Q associated with the nodes 1 and 2 respectively. Find explicit expressions for $N_1(x, y)$ and $N_2(x, y)$.

(c) Consider the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx$$

Compute the contribution of element Q to the matrix entry $A_{1,2} = a(\varphi_1, \varphi_2)$.

3 + 2 + 3 = 8 Points

Name: Matriculation-Nr.:

Aufgabe 5.

We consider the homogeneous shallow flow equations given as follows:

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\Big(\alpha hu^2 + \frac{1}{2}\left(gh^2\right)\Big) = 0, \end{cases}$$

with g being a positive parameter, so-called gravitational acceleration and α being a shape parameter, a real number, that compensates for a velocity profile. We write the two equations as a system

$$\partial_t U + \partial_x F(U) = 0.$$

a) Define U, and $F(U)$.

b) We linearize the given system as the following expression

$$\partial_t U + A(U)\partial_x U = 0.$$

Compute the flux *Jacobian* $A(U) = DF(U)$.

c) What are the characteristic velocities of the system?

d) Discuss under which condition (α, h, u) is the system hyperbolic?

1 + 2 + 2 + 3 = 8 Points

Name: Matriculation-Nr.:

Aufgabe 6.

For $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, consider the following system of linear conservation laws

$$\partial_t U(x,t) + A\partial_x U(x,t) = 0,$$

with $A := \begin{pmatrix} 0 & -2 \\ -1 & 1 \end{pmatrix}$ and the initial condition is given as follows

$$U(x,0) = \begin{cases} \begin{pmatrix} 0 \\ 4 \end{pmatrix}, & x < 0, \\ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, & x > 0. \end{cases}$$

Derive the solution by performing the following steps:

a) Diagonalise the system and transform the initial condition to reduce it to two independent scalar linear conservation laws by performaing variable transformation. Regarding this purpose use the eigenvalue decomposition of the matrix A as

$$A = T\Lambda T^{-1},$$

with Λ being diagonal matrix of eigenvalues.

b) Solve the acquired independent scalar conservation laws with the use of the corresponding initial conditions.

c) Transform the solutions back to the variable $U(x,t)$.

3 + 1 + 3 = 7 Points

Name: Matriculation-Nr.:

Aufgabe 7.

Consider the following initial value problem

$$\begin{aligned} u_t + a\,u_x &= 0 \qquad \text{for } t > 0, x \in \mathbb{R}, \\ u(0,x) &= u_0(x) \qquad \text{for } x \in \mathbb{R}. \end{aligned} \tag{I}$$

We use Upwind numerical scheme to solve the Eq. I. Further, we introduce cell-averaged value u_j^n,

$$u_j^n \approx \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u\left(t^n, x\right)\, dx,$$

on an equidistant grid with $t^n = n\Delta t$ and $x_j = j\Delta x$ where $n \in \mathbb{N}_0$ and $j \in \mathbb{Z}$.

a) Write down the update rule (u_j^{n+1}) for the Upwind numerical scheme

b) Define and introduce the CFL number. Rewrite the time-update rule $H(u_{j-1}, u_j, u_{j+1})$ as discussed in the lecture.

c) Determine the consistency order of the numerical scheme.

d) Derive a modified equation for the Eq. I.

1 + 2 + 2 + 2 = 7 Points

Name: Matriculation-Nr.:

Aufgabe 8.
Let Čada-Schmidtmann-Limiter be

$$\phi_{\check{C}S}(\theta) = \max\left(0, \min\left(\phi_3(\theta), \max\left(-\theta, \min\left(2\theta, \phi_3(\theta), 1.5\right)\right)\right)\right),$$

where $\phi_3(\theta) = \dfrac{2+\theta}{3}$.

a) Draw the ϕ-θ diagram of the above limiter for $\theta \in [-4, 4]$.

b) Why Čada-Schmidtmann-Limiter is not TVD?

c) What are the consequences of the above limiter not satisfying TVD condition?

3 + 1 + 2 = 6 Points

Name: Matriculation-Nr.: