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Öffnen Sie den Klausurbogen erst nach Aufforderung!

Partial Differential Equations (CES+SISC) | SS 2021
Klausur | 18. September 2021

Zugelassene Hilfsmittel:

- Dokumentenechtes Schreibgerät, aber kein Rotstift.
- Zwei eigenhändig und beidseitig beschriebene DIN A4 Blätter, die mit Namen und Matrikelnummer versehen sind.
- Weitere Hilfsmittel, insbesondere die Nutzung eines Taschenrechners, sind nicht erlaubt.

Hinweise:

- Das Mitführen von Mobilfunkgeräten während der Klausur gilt als Täuschungsversuch.
- Sie haben insgesamt **150 Minuten** Zeit zur Bearbeitung. *Alle Antworten sind ausführlich zu begründen*.
- Zum Bestehen der Klausur reichen **50%** der möglichen Punkte.
- Die Klausureinsicht findet am 08. Oktober 2021 von 14:00–15:00 Uhr im Seminarraum 328 (3. Stock) vom Rogowski-Gebäude (1090|328), Schinkelstraße 2 statt. Termine zur mündlichen Ergänzungsprüfung sind während der Klausureinsicht zu vereinbaren.
- Bitte beginnen Sie jede Aufgabe auf dem Blatt, auf dem die Aufgabenstellung formuliert ist. Sollten Sie außer der gegenüber befindlichen Leerseite noch eines der angehefteten Leerblätter benutzen, so geben Sie bitte auf dem ersten Blatt den Hinweis „Fortsetzung auf einem anderen Blatt“ an. *Bitte kennzeichnen Sie jedes Blatt mit Ihrem Namen und Ihrer Matrikelnummer – auch die benutzten Blanko-Blätter*.
- Durch Ihre Unterschrift versichern Sie, dass Sie zu Beginn der Klausur nach bestem Wissen prüfungsfähig sind und dass die Prüfungsleistung von Ihnen ohne nicht zugelassene Hilfsmittel erbracht wurde.

Matrikelnummer: ___ ___ ___ ___ ___ ___

Name, Vorname: ____________________

Unterschrift: ____________________

Aufgabe	1	2	3	4	5	6	7	8	$\sum$
Punkte	6	9	10	5	8	8	6	8	60
Ihre Punkte									

Klausur [ ] + Bonus [ ] = Gesamt [ ]

Note: [ ]

Aufgabe 1.

For each of the following functions check whether it is in $H^1([-1,1])$ or not. Justify your answers. Calculate the weak derivative for the functions that are in $H^1([-1,1])$.

(a)

$$u_1(x) = \max\{0, x\}$$

(b)

$$u_2(x) = \begin{cases} -\pi, & x < 0, \\ \pi, & x \geq 0. \end{cases}$$

(c)

$$u_3(x) = \sqrt{|x|}$$

2 + 2 + 2 = 6 Points

Aufgabe 2.

Let $\Omega \subset \mathbb{R}^d$ be an open bounded Lipschitz domain. Consider the following boundary value problem: Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u + \alpha u & = f \quad \text{in } \Omega \\ \nabla u \cdot \mathbf{n} + \beta u & = 0 \quad \text{on } \partial\Omega \end{cases} \tag{$*$}$$

for $f \in L^2(\Omega)$ given.

(a) Derive the weak formulation of the above problem. Also state the functions spaces.

(b) Find the values of α and β for which a unique weak solution exists.

(c) Suppose $\alpha, \beta \in \mathbb{R}$ are such that $(*)$ has a unique weak solution. Let u denote the unique solution of $(*)$. Suppose further that w is the unique weak solution of $(*)$ for $\tilde{f} \in L^2(\Omega)$ instead of f. Give an upper bound for the discrepancy $\|u - w\|_{H^1(\Omega)}$ in terms of the difference $\|f - \tilde{f}\|_{L^2(\Omega)}$.

3 + 4 + 2 = 9 Points

Aufgabe 3.

We consider the $1D$ boundary problem: Find u, such that,

$$\begin{cases} -\frac{\partial}{\partial x}\left((1+x^2)\frac{\partial}{\partial x}u(x)\right) = \sin(\pi|x|) & \text{for } x \in (-1,1) \\ u(-1) = u(1) = 0. \end{cases} \qquad (*)$$

(a) Of which of the following spaces is the weak solution an element of?

$$L^2([-1,1]),\ H^1([-1,1])$$

Hint: Of which of the spaces is the right hand side an element of? Is the differential operator continuous and coercive?

(b) We now want to discretize $(*)$ with the Galerkin method and derive the corresponding linear system as the matrix-vector equation

$$A\mathbf{c} = \mathbf{y},$$

with $A \in \mathbb{R}^{N\times N}$ and $\mathbf{c}, \mathbf{y} \in \mathbb{R}^N$.

Let V be a Hilbert space such that the bilinear form $a : V \times V \to \mathbb{R}$ as well as the linear form $b : V \to \mathbb{R}$ corresponding to $(*)$ satisfy the conditions of the Lax-Milgram theorem. We use the finite dimensional ansatz space

$$V_N = \{\Phi \in C^0(-1,1) | \Phi_{[x_i,x_{i+1}]} \in P_1,\ \Phi(-1) = \Phi(1) = 0\}$$

with $x_i = i/N$ for $i = -N, ..., N-1$ and P_1 the space of polynomials of degree $p \leq 1$. We choose hat functions ϕ_i as basis of V_N.

(i) State the definitions of the bilinear form a and the linear form b.

(ii) State the definitions of the Ritz-Galerkin solution $u_N \in V_N$, the matrix A and the two vectors $\mathbf{c}$ and $\mathbf{y}$ with respect to the chosen basis.

(iii) Calculate all entries of the matrix A.

Hint: Sketch a hat function ϕ_i and its two neighbours ϕ_{i-1}, ϕ_{i+1}.

(c) What order of convergence towards the exact solution do you expect for each of the following norms?

$$L^2(-1,1),\ H^1(-1,1),\ L^\infty(-1,1)$$

Explain your answer.

2 + (1+2+3) + 2 = 10 Points

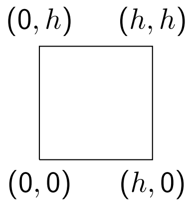
Aufgabe 4.

We consider in $2D$ a grid consisting of axially parallel squares with edge length h. On each square the ansatz function is given by

$$p(x, y) = \alpha + \beta x + \gamma y + \delta xy, \qquad (*)$$

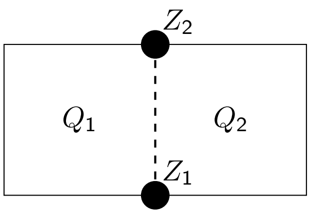
with $\alpha, \beta, \gamma, \delta, x, y \in \mathbb{R}$.
We define the degrees of freedom as the values of the ansatz functions at the corners of the square.

(a) Show, that the degrees of freedom are a family of unisolvent functionals, meaning that for the square



one can construct a unique ansatz function from the four function values at the edges.

(b) We now want to show, that the finite element space spanned by our chosen ansatz functions only contains continuous functions. To do that, consider two neighbouring squares Q_1 and Q_2, which share the edge E_{12} between the two corners Z_1 and Z_2 marked as dashed line below.



Further, let p_1 and p_2 be ansatz functions of form $(*)$ in Q_1 and Q_2, respectively, whose values are equal at the two corners Z_1 and Z_2, i.e.

$$p_1(Z_1) = p_2(Z_1) \text{ and } p_1(Z_2) = p_2(Z_2).$$

Show, that the values of these two ansatz functions are equal for *every* point on the edge E_{12} between Z_1 and Z_2, i.e. show:

$$\forall x \in E_{12} : p_1(x) = p_2(x).$$

2.5 + 2.5 = 5 Points

Aufgabe 5.

For $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ consider the equation

$$\partial_t u + \partial_x f(u) = 0 \tag{1}$$

a) Write down and simplify the integral formulation of Eq.(1) on the space-time domain $[a, b] \times [t_1, t_2] \subset \mathbb{R} \times \mathbb{R}^+, \ a < b, \ t_1 < t_2$.

b) Consider Eq.(1) with flux function

$$f(u) := u^2 + 3u \tag{2}$$

and initial condition

$$u(x, 0) = u_0(x) = \begin{cases} u_L & x < 0 \\ u_R & x > 0. \end{cases} \tag{3}$$

(i) Calculate the shock speed s. State and simplify the entropy condition.

Hint: Is the entropy $\eta(u)$ – entropy-flux $h(u)$ pair for Burgers' equation also a valid entropy – entropy-flux pair here?

(ii) State the condition on the values u_L, u_R for which the solution is a shock. State $u(x, t)$ for the shock solution.

(iii) State the condition on the values u_L, u_R for which the solution is a rarefaction wave. Calculate the solution $u(x, t)$ for this case.

(iv) For both cases identified in (ii) and (iii), state Godunov's flux $g(u_L, u_R)$.

1 + (2 + 1 + 2 + 2) = 8 Points

Aufgabe 6.

Consider a system of linear partial differential equation

$$\partial_t U + A\partial_x U = 0, \quad \text{in } \mathbb{R} \times (0, \infty)$$

with $A := \frac{1}{4}\begin{pmatrix} -2 & 3 \\ 12 & -2 \end{pmatrix}$ and the initial condition

$$U(x,0) = \begin{cases} \begin{pmatrix} 0 \\ 4 \end{pmatrix}, & x < 0, \\ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, & x > 0. \end{cases}$$

Find the solution to the above Riemann-Problem.

8 Points

Aufgabe 7.

The solution of the following Cauchy intial value problem

$$\begin{cases} u_t + au_x = 0 & \text{for } a > 0, t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{*}$$

is approximated by the following numerical scheme:

$$u_i^{j+1} = u_i^j - \frac{a\,k}{h}\left(u_i^{j+1} - u_{i-1}^{j+1}\right),$$

with

$$u_i^j \approx \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(t^j, x)\,dx,$$

on an equidistant grid $t^j = jk$, and $x_i = ih$.
Notice that the above scheme is **implicit**.

(a) Check whether or not the scheme is conservative.

(b) Determine the consistency order of the scheme.

(c) Derive a modified equation for the Equation (*) above.

2 + 2 + 2 = 6 Points

Name: Matrikel-Nr.:

Aufgabe 8.

Let $u_{i-1},\ u_i,\ u_{i+1}$ be three cell averages at a fixed time instance of an approximation to a conservation law on the space-time domain $\mathbb{R} \times \mathbb{R}^+$. Consider the following limiter reconstruction relations for the interface values

$$u^+(u_{i-1}, u_i, u_{i+1}) = u_i + \frac{1}{2}\Phi(\theta_i)(u_i - u_{i-1}) \ ,$$
$$u^-(u_{i-1}, u_i, u_{i+1}) = u_i - \frac{1}{2}\Phi(\theta_i)(u_i - u_{i-1})$$

with a limiter function Φ and $\theta_i = \dfrac{u_{i+1} - u_i}{u_i - u_{i-1}}$.

a) Show that for the limiter choice $\Phi(\theta) = \dfrac{1+\theta}{2}$, the reconstruction relations are given by $u^\pm = u_i \pm \dfrac{\Delta x}{2} D_x u|_i$, where $D_x u|_i$ is a spacial central difference approximation.

b) Let $(u_{i-1}, u_i, u_{i+1}) = (a, b, c)$. Due to symmetric constraints one has $u^+(a,b,c) = u^-(c,b,a)$. Show that $\Phi(\theta) = \theta\ \Phi\left(\dfrac{1}{\theta}\right)$ for $\theta > 0$.

c) Let the limiter function be $\Phi(\theta) = \min\left(\alpha\theta, \dfrac{1+\theta}{2}, \beta\right)$ with $\alpha, \beta \geq 1$. Sketch the limiter function for $\theta > 0$. Given the symmetric relation $\left(\Phi(\theta) = \theta\ \Phi\left(\dfrac{1}{\theta}\right)\right)$ from part b), what is the relation between α and β ?

2.5 + 2 + 3.5 = 8 Points