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Öffnen Sie den Klausurbogen erst nach Aufforderung!

Partial Differential Equations (CES+SISC) | SS 2020
Klausur | 07. October 2020

Zugelassene Hilfsmittel:

- Dokumentenechtes Schreibgerät, aber kein Rotstift.
- Zwei eigenhändig und beidseitig beschriebene DIN A4 Blätter, die mit Namen und Matrikelnummer versehen sind.
- Weitere Hilfsmittel, insbesondere die Nutzung eines Taschenrechners, sind nicht erlaubt.

Hinweise:

- Das Mitführen von Mobilfunkgeräten während der Klausur gilt als Täuschungsversuch.
- Sie haben insgesamt **150 Minuten** Zeit zur Bearbeitung. *Alle Antworten sind ausführlich zu begründen.*
- Zum Bestehen der Klausur reichen **50%** der möglichen Punkte.
- Die Klausureinsicht findet am 14. October 2020 von 11:20 – 13:00 Uhr im (1090|321) EPH, Rogowski Gebäude, Schinkelstraße 2 statt. Termine zur mündlichen Ergänzungsprüfung sind während der Klausureinsicht zu vereinbaren.
- Bitte beginnen Sie jede Aufgabe auf dem Blatt, auf dem die Aufgabenstellung formuliert ist. Sollten Sie außer der gegenüber befindlichen Leerseite noch eines der angehefteten Leerblätter benutzen, so geben Sie bitte auf dem ersten Blatt den Hinweis „Fortsetzung auf einem anderen Blatt“ an. *Bitte kennzeichnen Sie jedes Blatt mit Ihrem Namen und Ihrer Matrikelnummer – auch die benutzten Blanko-Blätter.*
- Durch Ihre Unterschrift versichern Sie, dass Sie zu Beginn der Klausur nach bestem Wissen prüfungsfähig sind und dass die Prüfungsleistung von Ihnen ohne nicht zugelassene Hilfsmittel erbracht wurde.

Matrikelnummer: ___ ___ ___ ___ ___ ___

Name, Vorname: ______________________

Unterschrift: ______________________

Aufgabe	1	2	3	4	5	6	7	8	$\sum$
Punkte	6	7	10	5	7	8	7	6	56
Ihre Punkte									

Klausur		Bonus		Gesamt
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Note:

Aufgabe 1.

For each of the following functions check whether it is in $H^1(\mathbb{R})$ or not. Justify your answers. Calculate the weak derivative for the functions that are in $H^1(\mathbb{R})$.

(a)

$$u_1 = \begin{cases} \sin(x), & x \in (-2\pi, 0) \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$u_2 = \begin{cases} \cos(x), & x \in (-\frac{3}{2}\pi, 0) \\ 0, & \text{otherwise} \end{cases}$$

(c)

$$u_3 = \begin{cases} \sin(x), & x < 0 \\ 0, & \text{otherwise} \end{cases}$$

2+2+2 = 6 Punkte

Name: Matrikel-Nr.:

Aufgabe 2.

Let $\Omega \subset \mathbb{R}$ be an open bounded domain. Consider the following Dirichlet boundary value problem

$$\begin{aligned} -\Delta u + \alpha u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

where $u \in H^1(\Omega)$ and $f, g \in L^2(\Omega)$.

(a) Derive the weak formulation of the above problem.

(b) Find the values of α for which there exists a unique solution.

3+4 = 7 Punkte

Aufgabe 3.

Consider the boundary value problem

$$\frac{d^4u}{dx^4} = f \qquad a < x < b$$
$$u(a) = u'(a) = 0$$
$$u(b) = u'(b) = 0$$

(a) Show that this problem satisfies the variational formulation

$$\int_a^b u''(x)v''(x)dx = \int_a^b f(x)v(x)dx \quad \forall v \in W$$

with $W = \{u \in H^2(a,b),\ u(a) = u'(a) = u(b) = u'(b) = 0\}$.

(b) On the reference interval $[0,1] \subset \mathbb{R}$ consider the polynomial space

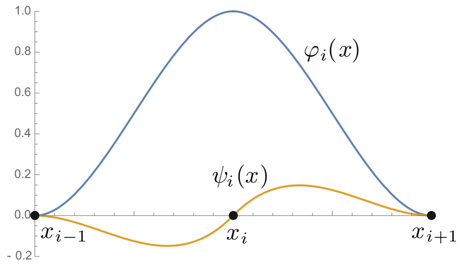
$$P_3 = \{p : \xi \mapsto p(\xi) \quad \text{polynomial}, \quad \deg(p) \leq 3\}.$$

What is the dimension of this space? Give a basis of this space by constructing form functions $\hat{N}_i(\xi)$, $i = 1,2,3,4$ such that each form function gives one for exactly one of the values $\{N_i(0),\ N_i(1),\ N_i'(0),\ N_i'(1)\}$ and zero for the other three.

(c) For an interval $\Omega = [a,b]$ we consider the positions $\{x_i\}_{i=0,\dots N+1} \subset [a,b]$ with $a = x_0 < x_i < x_{N+1} = b$. Lets define the intervals $K_i = [x_i, x_{i+1}]$ as elements and define the differentiable finite-element space

$$\hat{S}^3\left([a,b], \{K_i\}_{i=1}\right) = \left\{u \in C^1([a,b]),\ u(x)|_K = \text{polynomial with degree } 3\right\}.$$

As basis for this space we use hat functions φ_j such that at the points $\{x_i\}_{i=0,\dots N+1}$ we have $\varphi_j(x_i) = \delta_{ij}$ and $\varphi_j'(x_i) = 0$ for the derivatives, **as well as** shape functions ψ_j such that $\psi_j'(x_i) = \delta_{ij}$ and $\psi_j(x_i) = 0$



What is the dimension of $\hat{S}^3$ for $N+1$ elements?

(d) Let us choose $a = -1$ and $b = 1$ and consider the above boundary value problem with a discretization of the intervall $[a,b]$ by two elements $K_0 = [-1,0]$ and $K_1 = [0,1]$ each equipped with the space P_3. Using the basis from (c) and after exploiting the boundary values what are the two remaining basis functions of this discretization?

(e) Give a solution for the variational formulation of the boundary value problem and the discretization of (d) for the case $f(x) = 1$.

2+2+2+1+3 = 10 Punkte

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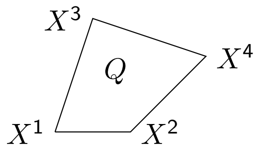
Aufgabe 4.

Transformation from the reference square $\widehat{Q} = [0,1]^2$ to a general quadrilateral Q given by vertices $X^i = (x_i, y_i)^\top$ for $i = 1,2,3,4$ can be obtained by the mapping F given by

$$F(\theta) = X^1(1-\psi)(1-\eta) + X^2\psi(1-\eta) + X^3(1-\psi)\eta + X^4\psi\eta,$$

with $\theta = (\psi, \eta)$. The order of vertices follows the scheme



(a) Compute $\nabla F(\theta)$.

(b) Show that determinant of the gradient $\det(\nabla F(\theta)) \geq 0$ if and only if the quadrilateral is convex.
Hinweis: $\det([X^2 - X^1, X^3 - X^1])$ is non-negative if counter-clockwise oriented angle $\angle X^2 X^1 X^3$ is in range $[0, \pi]$.

1+4 = 5 Punkte

Aufgabe 5.

For a following system of linear conservation laws

$$\partial_t U(t,x) + A\partial_x U(t,x) = 0,$$

with a matrix $A = \begin{pmatrix} -7 & 10 \\ 5 & -2 \end{pmatrix}$ and an initial condition

$$U(0,x) = \begin{cases} \begin{pmatrix} 0 \\ 4 \end{pmatrix}, & x < 0, \\ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, & x > 0. \end{cases}$$

Consider following steps:

(a) Diagonalise the system and transform the initial condition to reduce it to two independent scalar linear conservation laws by changing variables. For this purpose use the eigenvalue decomposition of the matrix $A = T\Lambda T^{-1}$ with a diagonal Λ.

(b) Solve acquired independent scalar conservation laws with corresponding initial conditions.

(c) Transform the solution back to the variable $U(t,x)$.

3+1+3 = 7 Punkte

Aufgabe 6.

Consider the initial value problem

$$u_t + \left(\frac{(u-1)^2}{2}\right)_x = 0$$
$$u(x,0) = u_0(x) = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases}$$

(a) Specify the shock speed s and the entropy condition,

(b) Specify values u_L, u_R for which the solution is a shock and specify the solution,

(c) Specify values u_L, u_R for which the solution is a rarefaction wave and specify the solution,

(d) For both cases present Godunov's flux $g(u_L, u_R)$.

2+1+2+3 = 8 Punkte

Aufgabe 7.

Consider the following advection equation

$$\partial_t u + a\partial_x u = 0 \qquad \forall x \in \Omega,$$

where $a \in \mathbb{R}^+$. We will discretize the above equation using finite differences and a upwind scheme in the following way

$$u_i^{n+1} = u_i^n - a\lambda(u_i^n - u_{i-1}^n).$$

Here u_i^n represents the solution at the i-th grid point and the n-th time step. The factor λ is the ratio of Δt and Δx i.e. $\lambda = \Delta t/\Delta x$. Show that

(a) The above upwind scheme is consistent upto at least first order.

(b) Let C be the amplification factor of the scheme, then

$$C^2 = (1 - \lambda a + \lambda a \cos(\theta))^2 + (\lambda a \sin(\theta))^2$$

where $\theta = k\Delta x$ with k being the wave number of the Fourier ansatz.

(c) For what ranges of λ is the scheme stable.

3+2+2 = 7 Punkte

Aufgabe 8.

Assume the following Čada-Schmidtmann-Limiter

$$\phi_{\check{C}S}(\theta) = \max\left(0, \min\left(\phi_3(\theta), \max\left(-\theta, \min\left(2\theta, \phi_3(\theta), 1.5\right)\right)\right)\right),$$

with $\phi_3(\theta) = \dfrac{2+\theta}{3}$.

(a) Draw the limiter for $\theta \in [-4, 4]$ in the ϕ-θ-diagramm.

(b) Why is the Čada-Schmidtmann-Limiter not TVD?

(c) What benefit could the violation of the TVD condition have?

4+1+1 = 6 Punkte

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